189-346/377B: Number Theory
Assignment 1

Due: Monday, January 21

The following questions are designed to make you familiar with a computer algebra system and also to give you some practice in manipulating real and complex numbers on Pari, and testing their algebraicity.

1. Compute $e^{\pi\sqrt{163}}$ with 30 significant digits on Pari. (For this, first enter the Pari command $\backslash p$ 30. Note that the constant π in Pari is written `Pi` (with a capital p) and that the square root and exponential functions are `sqrt` and `exp`.) What do you observe? Repeat the calculation with 40 significant digits. (This exercise is meant to get you familiar with using Pari, and also as a cautionary tale about drawing conclusions too hastily concerning rationality or algebraicity based on experimental data.)

2. Pari has a wonderful command called `algdep`, which takes as input a real number α (computed to some accuracy) and an integer d, and attempts to find a polynomial of degree d satisfied by α, whose coefficients are as small as possible. To get a feeling of how it works, set your accuracy to 500 significant digits, and type `algdep(Pi, j)` for j running from 1 to 8. This calculation doesn't prove anything about π but it does provide *experimental evidence* suggesting that π is trancendental (or at least, that π is not the root of a small polynomial of degree $\leq$ 8!) Do the same calculation with the real number

$$\alpha = \sqrt{6} + 2\sqrt{\sqrt{3}+\sqrt{2}} + \sqrt{\sqrt{3}-\sqrt{2}},$$

and write down the unique monic polynomial of minimal degree satisfied by α.

3. Compute $q = e^{-2\pi\sqrt{5}}$ and

$$\alpha = q^{-1} \prod_{n=1}^{\infty} (1+q^n)^{-24}$$

to 200 digits of real accuracy using PARI. Try recognising α as an algebraic number by typing $\mathtt{algdep}(\alpha, \mathtt{d})$ for $1 \leq d \leq 6$. What do you observe? Formulate a hunch, and test it by performing the same calculation to 500 digits of accuracy. Repeat the same exercise (working this time with an accuracy of 500 and then 1000 decimal digits) with $q = e^{-2\pi\sqrt{23}}$. (For this question, you might find the Pari commands `exp`, `sqrt`, and `prod` useful. Note that you can find out more about about a given command, say `algdep`, by typing `?algdep` from the Pari command prompt.)

Cultural remark. The patterns observed in questions 1 and 3 are striking and actually *proving* (as opposed to merely testing experimentally) the implied algebraicity statements involves sophisticated mathematical ideas (related to *modular forms*, and the theory of *complex multiplication of elliptic curves*) going well beyond the standard undergraduate curriculum. The explanation turns out to involve in a crucial way the fact that $\mathbf{Z}\left[\frac{1+\sqrt{-163}}{2}\right]$ has unique factorisation, while $\mathbf{Z}[\sqrt{-5}]$ and $\mathbf{Z}\left[\frac{1+\sqrt{-23}}{2}\right]$ do not!

4. Let K be a number field. A subring R of K (i.e., a subset which is closed under both addition and multiplication, but not necessarily under division) is said to be *of finite type* if there exists a finite set $e_1, \ldots, e_n$ of elements of R, such that every element of R can be written as a linear combination of $e_1, \ldots, e_n$ with *integer* coefficients. Show that the following properties of $\alpha \in K$ are equivalent:

- The number α is integral, as defined in class, i.e., α is the root of a monic polynomial with integer coefficients. (One also says that α is an *algebraic integer*.)
- The ring
$$\mathbf{Z}[\alpha] := \{f(\alpha), \quad f(x) \in \mathbf{Z}[x]\} \subset K$$
generated by α and all its powers is of finite type.

- The number α is contained in a subring R of K which is of finite type.

(Hint: it is easiest to to show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$, the last implication being the most tricky. To show it, use 3 to represent α by a matrix with integral entries and apply the Cayley-Hamilton theorem.)

5. If α and β are integral elements of K, show that the ring

$$\mathbf{Z}[\alpha, \beta] = \{f(\alpha, \beta) \quad f(x, y) \in \mathbf{Z}[x, y]\}$$

is s subring of K of finite type. Using exercise 4, conclude that $\alpha + \beta$ and $\alpha\beta$ are both integral, and therefore that the set $\mathcal{O}_K$ of integral elements of K is closed under addition and multiplication. (And hence, it forms a *subring* of K, just like the usual integers $\mathbf{Z}$ are a subring of the rational numbers $\mathbf{Q}$.)

6. Using what you've learned in questions 4 and 5, show that α is an algebraic integer if and only if the (unique) monic polynomial of minimal degree satisfied by α has integer coefficients. (This is the definition of algebraic integer which I relied on in the class lectures in calculating various rings of algebraic integers and showing, for example, that $\mathbf{Z}[i]$ is the ring of integers of $\mathbf{Q}(i)$.)