Étale and Pro-Unipotent fundamental groups: Lecture 1

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Motivations

Let $E/\mathbb{Q}$ be an elliptic curve over $\mathbb{Q}$. In previous lectures we have highlighted the importance of the **Kummer exact sequence**:

$$0 \longrightarrow E(\mathbb{Q})/nE(\mathbb{Q}) \longrightarrow H^1(G_{\mathbb{Q}}, E[n]) \longrightarrow H^1(G_{\mathbb{Q}}, E(\overline{\mathbb{Q}}))[n] \longrightarrow 0 \tag{1}$$

in the study of $E(\mathbb{Q})$. For example, we can prove that $E(\mathbb{Q})$ is finitely generated (**Mordell-Weil Theorem**) by proving that the left-hand term of (1) is finite. This is not immediately clear from (1), since $H^1(G_{\mathbb{Q}}, E[n])$ is infinite in general, but the situation can be remedied by imposing local conditions at the primes dividing n and at the primes of bad reduction. We can then refine (1) to obtain an injection:

$$E(\mathbb{Q})/nE(\mathbb{Q}) \hookrightarrow H^1_f(G_{\mathbb{Q}}, E[n]) \tag{2}$$

into a **Selmer group** which will always be finite, giving Mordell-Weil. Moreover, the cokernel of this injection is closely related to the **Birch and Swinnerton-Dyer conjecture**, and any understanding of it should translate into some progress towards the full BSD. A detailed account of this construction, together with several examples, can be found in [Sil85].

Generalizing (2) to a smooth projective algebraic curve $X/\mathbb{Q}$ of higher genus $g > 1$ immediately presents some difficulties. For starters, there is no group law on X that is compatible with its structure of algebraic variety, hence there are no 'multiplication-by-n' maps available. We can attempt to circumvent this problem by working with the **Jacobian variety** J_X associated to X, an abelian variety of dimension g defined over $\mathbb{Q}$, and a chosen embedding $\iota : X \hookrightarrow J_X$. One can then attempt to study the injection:

$$J_X(\mathbb{Q})/nJ_X(\mathbb{Q}) \hookrightarrow H^1(G_{\mathbb{Q}}, J_X[n]) \tag{3}$$

by imposing local conditions, and hope to gain information about the structure of $X(\mathbb{Q})$ via the embedding $X(\mathbb{Q}) \hookrightarrow J_X(\mathbb{Q})$. This approach, for example, was successfully employed by Mazur in [Maz77] to compute the $\mathbb{Q}$-rational points of the modular curves $X_1(N)$ and $X_0(N)$. Knowledge of these rational points gives a complete classification of the rational torsion structures occuring in elliptic curves over $\mathbb{Q}$.

The Jacobian approach, however, has no hope of giving satisfactory answers for general curves $X/\mathbb{Q}$: taking Jacobians in fact kills a huge part of the arithmetic-geometric structure of X. This phenomenon is perhaps easier to explain at the topological level first. If we extenend scalars to $\mathbb{C}$, the complex points $X(\mathbb{C})$ have the structure of a compact complex manifold, and $J_X(\mathbb{C})$ has the structure of a torus $\mathbb{C}^n/\Lambda$. Now the fundamental group of $X(\mathbb{C})$ is the free group on $2g$ generators modulo some 'gluing' conditions, and it is not abelian as soon as $g > 1$. However, it is not hard to show that:

$$\pi_1(\mathbb{C}^n/\Lambda; 0) \simeq \Lambda \simeq \pi_1(X(\mathbb{C}); b)^{\text{ab}},$$

hence all the 'non-abelian' topological information contained in $\pi_1(X(\mathbb{C}); b)$ is lost when taking Jacobians.

A similar phenomenon occurs at the algebraic level. First, we package all the exact sequences (3) into an inverse system, and take the limit:

$$\widehat{J_X(\mathbb{Q})} \longrightarrow H^1(G_{\mathbb{Q}}, T(J_X))$$

where $T(J_X) = \prod_\ell T_\ell(J_X)$ is the full Tate module of J_X. The theory of the **étale fundamental group**, pioneered by Grothendieck in [Gro61], constructs for any algebraic variety an analog π_1^{et} of the topological fundamental group. For J_X, and indeed for any abelian variety, we recover the Tate module: if we let

$$\overline{J_X} = J_X \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\overline{\mathbb{Q}}$$

then

$$T(J_X) \simeq \pi_1^{\text{et}}(\overline{J_X}; o). \tag{4}$$

But $\pi_1^{\text{et}}(\overline{X}; b)$ is a genuinely new, non-abelian invariant of X, for which we have:

$$\pi_1^{\text{et}}(\overline{X}; b)^{\text{ab}} \simeq \pi_1^{\text{et}}(\overline{J_X}; o)$$

just as in the topological case. Therefore, whereas for elliptic curves (which have abelian fundamental group) the Kummer map (1) should suffice to understand rational points, for a general curve X of higher genus the map (3) does not seem to capture all the structure of $X(\mathbb{Q})$.

The goal of this lecture is to explain how to replace (1) and (3) by the **étale period map**:

$$X(\mathbb{Q}) \longrightarrow H^1(G_{\mathbb{Q}}, \pi_1^{et}(\overline{X}; b)) \tag{5}$$

for curves X of higher genus. The fact that $\pi_1^{et}(\overline{X}; b)$ is not abelian places this map in the realm of **non-abelian group cohomology**, a topic covered in the last lecture. In particular, by the non-abelian cohomology philosophy, the set on the right of (5) can be interpreted as the set of **torsors** under $\pi_1^{et}(\overline{X}; b)$. The **section conjecture** of Grothendieck (Conjecture 1.24 below) states that the map (5) is a bijection for curves of genus $g > 2$. A proof of this conjecture seems today to be beyond the reach of our current techniques.

The goal of the next lecture will be to setup Minhyong Kim's program, which replaces (5) with more tractable analogs. In particular, we will construct the **$\mathbb{Q}_p$-pro-unipotent completion of** $\pi_1^{\text{et}}(\overline{X}; b)$, which will be denoted by $\pi_1^{u,\mathbb{Q}_p}(\overline{X}; b)$, and define the **unipotent period map**:

$$X(\mathbb{Q}) \longrightarrow H^1(G_{\mathbb{Q}}, \pi_1^{u,\mathbb{Q}_p}(\overline{X}; b)). \tag{6}$$

One of the advantages of replacing $\pi_1^{et}(\overline{X}; b)$ with $\pi_1^{u,\mathbb{Q}_p}(\overline{X}; b)$, as we shall see, is that the action of $G_{\mathbb{Q}}$ on $\pi_1^{u,\mathbb{Q}_p}(\overline{X}; b)$ can be computed in terms of the action on $T(J_X)$, a feature not available in the full étale case. Moreover, as we will see in future lectures, we can impose local conditions at p and at the primes of bad reduction to obtain a map

$$X(\mathbb{Q}) \longrightarrow H^1_f(G_{\mathbb{Q}}, \pi_1^{u,\mathbb{Q}_p}(\overline{X}; b)) \tag{7}$$

where the right-hand side acquires the structure of a pro-algebraic variety, a so-called **Selmer variety**. In a way, this can be seen as a unipotent, non-abelian substitute for the injection $X \hookrightarrow J_X$, and for this reason Kim calls (7) the **unipotent Albanese map**. The study of this map, despite it being a coarse approximation of the étale period map (5), already yields important results about rational points on hyperbolic curves, as we shall see in detail in future lectures.

1 Fundamental groups: the algebraic theory

1.1 The topological π_1

Traditionally, the first construction encountered in algebraic topology is the functor

$$\pi_1 : (\mathbf{Top}; *) \longrightarrow \mathbf{Gps}$$

from the category of pointed topological spaces to the category of groups. This functor is constructed by attaching to the pair $(X;b)$ the group of homotopy classes of loops based at b. One soon learns that this geometrically intuitive definition is unsuitable for computations even in the simplest non-trivial examples, such as that of the circle S^1. Instead, the subject acquires its depth by its interplay with the theory of covering spaces.

Let $\mathbf{Cov}(X)$ be the category of covering spaces of X, whose morphisms are continuous maps over X, and consider the **fiber functor**:

$$\begin{aligned} F_b : \mathbf{Cov}(X) &\longrightarrow \mathbf{Sets} \\ \{f : Y \mapsto X\} &\longmapsto f^{-1}(b). \end{aligned}$$

THEOREM 1.1. *Suppose X is connected, locally path-connected and semi-locally simply connected. Then the functor F_b is representable, i.e. there exists a covering space $\widetilde{X}_b \in \mathbf{Cov}(X)$ such that:*

$$F_b = \mathrm{Hom}(\widetilde{X}_b, -)$$

as functors $\mathbf{Cov}(X) \to \mathbf{Sets}$.

Proof. [Hat02] Proposition 1.38. □

We call $\widetilde{X}_b$ the **universal covering space** of $(X;b)$. As a set, $\widetilde{X}_b$ is constructed as the set of homotopy classes of paths in X starting at b. The covering map is given by:

$$\begin{aligned} \tilde{f} : \widetilde{X}_b &\longrightarrow X \\ [\gamma] &\longmapsto \gamma(1) \end{aligned}$$

so that the fiber $\tilde{f}^{-1}(x)$ can be identified with $\pi_1(X;b,x)$. In particular, the fiber above b is precisely $\pi_1(X;b)$. From this description we see that the natural right action of $\pi_1(X;b)$ on $\widetilde{X}_b$, given by pre-composition of paths, induces a map

$$\pi_1(X;b) \longrightarrow \mathrm{Aut}^{\mathrm{op}}_{\mathbf{Cov}(X)}(\widetilde{X}_b) \tag{8}$$

since the action respects the fibers of $\widetilde{f}$.

THEOREM 1.2. *The map (8) is an isomorphism.*

Proof. [Hat02] Proposition 1.39. □

In particular, we deduce that $\widetilde{X}_b$ is simply connected: it must be its own universal covering space by the universal property, hence

$$\pi_1(\widetilde{X}_b;\widetilde{b}) = \mathrm{Aut}^{\mathrm{op}}_{\mathbf{Cov}(\widetilde{X}_b)}(\widetilde{X}_b) = \{\mathrm{id}\}.$$

Conversely, any simply connected covering $(Y;a) \to (X;b)$ must be the universal covering space: the covering map $\widetilde{X}_b \to Y$ has degree $|\pi_1(Y;a)| = 1$, hence it is a homeomorphism. This criterion translates the problem of computing fundamental groups to that of computing simply connected covering spaces, and can be used to compute fundamental groups in many basic instances.

Example 1.3. The punctured complex plane $(\mathbb{C}^\times;1)$ has a covering map given by:

$$\begin{aligned} \exp : (\mathbb{C};0) &\longrightarrow (\mathbb{C}^\times;1) \\ z &\longmapsto e^z. \end{aligned}$$

Since $\mathbb{C}$ is simply connected, it is the universal covering space of $(\mathbb{C}^\times;1)$. Therefore

$$\pi_1(\mathbb{C}^\times;1) = \exp^{-1}(1) = 2\pi\mathrm{i}\,\mathbb{Z} \simeq \mathbb{Z}.$$

Example 1.4. The complex points $(A(\mathbb{C});o) \simeq (\mathbb{C}^n/\Lambda;0)$ of an abelian variety of dimension n have a covering map:

$$p : (\mathbb{C};0) \longrightarrow (\mathbb{C}/\Lambda;0).$$

Since $\mathbb{C}$ is simply connected,

$$\pi_1(A(\mathbb{C});o) = p^{-1}(0) = \Lambda \simeq \mathbb{Z}^{2n}.$$

Example 1.5. For the doubly-punctured plane $\mathbb{C} - \{0,1\}$ the construction of a universal covering space is a bit more involved... unless you are a number theorist. Consider in fact the congruence subgroup $\Gamma = \Gamma(2) \subset \mathbf{SL}_2(\mathbb{Z})$ of matrices which reduce to the identity modulo 2. This group acts on the complex upper half-plane via linear fractional transformations, and the action factors through $\mathbf{PSL}_2(\mathbb{Z})$. It is well-known that the quotient $\mathbb{H}/\Gamma$ is in bijection with pairs $(E,\{e_1,e_2\})$ of an elliptic curve over $\mathbb{C}$ together with a choice of basis for the full 2-torsion $E[2]$. On the other hand, such elliptic curves can always be written in the form:

$$y^2 = x(x-1)(x-\lambda), \qquad \lambda \in \mathbb{C} - \{0,1\}$$

by [Sil85] Proposition III.1.7. The composition:

$$\begin{aligned} \mathbb{H} \longrightarrow \quad \mathbb{H}/\Gamma \xrightarrow{\simeq} \mathbb{C} - \{0,1\} \\ (E, \{e_1, e_2\}) \longmapsto \lambda \end{aligned}$$

is a covering map, which can be written down explicitly in terms of modular functions for $\Gamma(2)$. Since $\mathbb{H}$ is simply connected, we must have:

$$\pi_1(\mathbb{C} - \{0,1\}; b) \simeq \overline{\Gamma} \subset \mathbf{PSL}_2(\mathbb{Z})$$

for any base point b. Now the group $\overline{\Gamma}$ is freely generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ so we have:

$$\pi_1(\mathbb{C} - \{0,1\}; b) \simeq \mathbb{Z} * \mathbb{Z}$$

as intuition dictates. Note that the same is true for $\mathbb{P}^1 - \{0, 1, \infty\}$.

In the examples above the groups π_1 were computed without any mention of paths: only the functorial properties of π_1 have been used. Therefore, it seems that π_1 itself should admit a category-theoretical description. To this end, note that by Theorem 1.1 we have:

$$\mathrm{Aut}(F_b) = \mathrm{Aut}(\mathrm{Hom}(\widetilde{X}_b, -))$$

and by Yoneda's lemma:

$$\mathrm{Aut}(\mathrm{Hom}(\widetilde{X}_b, -)) = \mathrm{Aut}(\widetilde{X}_b).$$

Combined with Theorem 1.2, we observe that:

$$\pi_1(X; b) = \mathrm{Aut}(F_b) \tag{9}$$

and by an entirely analogous reasoning:

$$\pi_1(X; b, x) = \mathrm{Isom}(F_b, F_x). \tag{10}$$

We have thus found an entirely functorial decription of path and loop spaces, which is the starting point for the algebraic theory of fundamental groups.

1.2 The étale π_1

Let now X/k be an algebraic variety over an aglebraically closed field k. Ideally we would like to define a functor:

$$(\mathbf{Var_k}; *) \longrightarrow \mathbf{Gps}$$

from the category of pointed varieties over k to the category of groups, which recovers the topological π_1 when evaluated on algebraic varieties over $\mathbb{C}$. Let us say right away that this is not possible, mainly because for a complex algebraic variety X the Zariski topology on $X/\mathbb{C}$ is of an entirely different nature than the analytic topology on $X(\mathbb{C})$. For example, in the algebraic setting the notion of path is not at all well-behaved (e.g. a path in an irreducible curve is a point). Moreover the notion of a local homeomorphism (let alone that of covering space) is too restrictive for the Zariski topology. For example the projection of the variety $X = \{y - x^2 = 0\} \subset \mathbb{C}^2$ onto the y-axis minus the origin, which is a covering map in the analytic topology, is not even a local homeomorphism in the Zariski topology: no Zariski open set of $\mathbb{C}^2$ can be chosen so that it intersects only one 'branch' of X.

Grothendieck's insight was to replace the notion of a covering space by that of **finite étale covers** (Definition 1.11 below) and then use the functorial description (9) of the fundamental group to avoid the use of paths. The result is a functor

$$\pi_1^{\text{et}} : (\mathbf{Var_k}; *) \longrightarrow \mathbf{PfGps}$$

from the category of pointed algebraic varieties over k to the category of profinite groups. When $k = \mathbb{C}$, the functor computes $\widehat{\pi_1}$, the profinite completion of the topological fundamental group. In other words, the profinite completion of the topological π_1 of a complex algebraic variety has a purely algebraic description!

The starting point for the definition of an algebraic notion of local isomorphism is to observe that the *inverse function theorem* for a morphism $f : X \to Y$ of complex algebraic varieties gives an entirely algebraic criterion to establish whether f is a local isomorphism. This observation motivates the following definition.

Definition 1.6. A morphism $f : X \to Y$ of varieties over an algebraically closed field k is *étale* if it is smooth of relative dimension 0. This means

(I) **Locally:** If f is given by

$$f : X = \operatorname{Spec}\left(\frac{A[X_1, \ldots, X_n]}{(p_1, \ldots, p_n)}\right) \longrightarrow \operatorname{Spec}(A) = Y$$

where A is a k-algebra, then f is étale if and only if

$$\det\left(\frac{\partial p_i}{\partial X_j}\right)(x) \in k^\times$$

for all $x \in X$.

(II) **Globally:** For all $x \in X$, there are open neighborhoods U of x and V of $f(x)$ such that $f(U) \subset V$ and $f|U$ comes from restriction of a morphism of type (I).

Example 1.7. The projection of $y = x^2$ onto the y-axis minus the origin is étale if $\mathrm{char}(k) \neq 2$.

Example 1.8. The map $x \mapsto x^n$ from $\mathbb{G}_{m/k}$ to itself is étale if $\mathrm{char}(k) \nmid n$.

Example 1.9. The multiplication-by-n map on an abelian variety A/k is étale $\mathrm{char}(k) \nmid n$.

Example 1.10. The projection of $\{xy - 1 = 0\}$ onto the x-axis is étale.

As a safety check, note that when $k = \mathbb{C}$ a morphism of complex varieties $f : X \to Y$ induces a local isomorphism for the analytic topology on the complex points $f : X(\mathbb{C}) \to Y(\mathbb{C})$, and vice-versa ([Mum99] III.5 Corollary 2).

The algebraic notion of a covering space is the following.

Definition 1.11. A morphism $f : Y \to X$ of algebraic varieties over an algebraically closed field k is a *finite étale* morphism if it is finite and étale. We call Y a *finite étale cover* of X.

Example 1.12. All the maps in Examples 1.7, 1.8 and 1.9 are finite étale.

Example 1.13. The map in Example 1.10 is not finite, hence not finite étale.

Remark 1.14. When $k = \mathbb{C}$ the concept of finite étale map coincides with that of a *finite covering map* for the analytic topology. In fact, while the *étale* requirement on f replaces the notion that a covering map must be a local homeomorphism, the *finite* requirement replaces the idea that a covering map should have at least one 'sheet' covering the entire base space (and no more than finitely many, for finite coverings). This is to avoid that open inclusions are coverings, as in Example 1.10, and ensure surjectivity.

Let now $\mathbf{FEtCov}(X)$ be the category of finite étale maps $f : Y \to X$ (morphisms are morphisms of k-varieties over X), and fix a closed point $b \in X(k)$. Define the **étale fiber functor**:

$$\begin{aligned} F_b^{\mathrm{et}} : \mathbf{FEtCov}(X) &\longrightarrow \mathbf{Sets} \\ \{f : Y \to X\} &\longmapsto f^{-1}(b) = \mathrm{Spec}(f_*(\mathcal{O}_Y)_b \otimes_{\mathcal{O}_{X,b}} k(b)). \end{aligned}$$

Inspired by (9), define the **étale fundamental group** of $(X; b)$ by

$$\pi_1^{\mathrm{et}}(X; b) := \mathrm{Aut}(F_b^{\mathrm{et}}) \tag{11}$$

and similarly

$$\pi_1^{\text{et}}(X; b, x) := \text{Isom}(F_b^{\text{et}}, F_x^{\text{et}}) \tag{12}$$

for any other closed point $x \in X(k)$.

REMARK 1.15. Similar definitions can be given for any pair $(S; b)$ of a scheme and a geometric point on it. This is necessary, for example, if one wants to study how $\pi_1^{\text{et}}(X; b)$ varies in a family of algebraic varieties. The ultimate reference for this general approach is SGA-1 [Gro61].

Just as in the topological case, definitions (11) and (12) are as conceptually satisfying as they are unworkable. We wish then to find an algebraic analog for the universal covering space $\widetilde{X}_b$, i.e. we wish to *represent* the étale fiber functor F_b^{et}. Intuitively, it is clear that this is not possible in general: we need to find a finite étale cover that dominates all others, but any two finite covers can be combined (fiber product) to give a finite cover that dominates both of them. This intuition suggests that the analog of a universal covering space for finite étale maps should be an inverse limit of finite covers, where the inverse sytem is constructed via fiber products. Of course, the resulting object will not be a variety, but only a **pro-variety**: we have landed upon the concept of a **pro-representable** functor.

DEFINITION 1.16. Let $\mathcal{C}$ be a category, and F a set-valued functor on $\mathcal{C}$. We say F is *pro-representable* if there exists an inverse system $P = (P_\alpha, \phi_{\alpha\beta})$ of objects of $\mathcal{C}$ indexed by a directed partially ordered set I, and a functorial isomorphism:

$$\lim_{\rightarrow} \text{Hom}(P_\alpha, X) \simeq F(X)$$

for each object X in $\mathcal{C}$.

THEOREM 1.17. *The étale fiber functor F_b^{et} is pro-representable.*

Proof. We construct the inverse system by taking *Galois* covers of X, i.e. connected covers $f_\alpha : X_\alpha \to X$ such that the group $\text{Aut}_X(X_\alpha)$ acts transitively on the fibers. These form a directed set, since for any two Galois covers X_α, X_β, we can form the fiber product $X_\alpha \times_X X_\beta$, a finite étale cover, and find a maximal finite étale cover of it which is Galois over X. Now there is no unique choice of morphisms $\phi_{\alpha\beta} : X_\beta \to X_\alpha$. We must require them to fix a choice of elements $\{b_\alpha \in f_\alpha^{-1}(b)\}$ for each α. The existence of such compatible system of morphisms is guaranteed by the Galois property. Moreover, once the choice of $\{b_\alpha \in f_\alpha^{-1}(b)\}$ is fixed, the compatible system is unique. For details, see [Sza09] Proposition 5.4.6. □

It is a formal consequence of the definitions that the étale fundamental group can be computed directly from the inverse system $\widetilde{X}_b = \{(X_\alpha; b_\alpha)\}_\alpha$.

Theorem 1.18. *Let X/k be an algebraic variety over an algebraically closed field k. Let $b \in X(k)$ be a closed point and let $\widetilde{X}_b = \{(X_\alpha; b_\alpha)\}_\alpha$ be the pro-algebraic variety of Theorem 1.17. If $\widetilde{f} = \{f_\alpha\}$ denotes the map $\widetilde{f} : \widetilde{X}_b \to X$, then:*

(a) $\pi_1^{\mathrm{et}}(X; b) = \widetilde{f}^{-1}(b)$.

(b) $\pi_1^{\mathrm{et}}(X; b, x) = \widetilde{f}^{-1}(x)$ *for any other point* $x \in X(k)$.

Proof. It suffices to prove (b), (a) being a special case. By Yoneda's lemma:

$$\mathrm{Nat}(F_b^{\mathrm{et}}, F_x^{\mathrm{et}}) = \mathrm{Hom}(\widetilde{X}_x, \widetilde{X}_b) = F_x(\widetilde{X}_b) = \widetilde{f}^{-1}(x).$$

Moreover, note that any element in $\mathrm{Hom}(\widetilde{X}_x, \widetilde{X}_b)$ must necesserily be invertible, by the universal properties of both $\widetilde{X}_x$ and $\widetilde{X}_b$, so that:

$$\mathrm{Nat}(F_b^{\mathrm{et}}, F_x^{\mathrm{et}}) = \mathrm{Isom}(F_b^{\mathrm{et}}, F_x^{\mathrm{et}}) = \pi_1^{\mathrm{et}}(X; b, x).$$

□

Example 1.19. Suppose $\mathrm{char}(k) = 0$ and consider $(\mathbb{G}_{m/k}; 1)$, where $\mathbb{G}_{m/k} = \mathrm{Spec}(k[t, t^{-1}])$. Then the maps $[n] : t \mapsto t^n$ give a compatible system $\widetilde{\mathbb{G}_{m/k}} = \{[n] : \mathbb{G}_{m/k} \to \mathbb{G}_{m/k}; 1\}_n$ of Galois covers, in the sense of Theorem 1.17. We then have

$$\pi_1^{\mathrm{et}}(\mathbb{G}_{m/k}; 1) = \widetilde{f}^{-1}(1) = \widehat{\mu} = \mathrm{proj\,lim}\,\mu_n \simeq \widehat{\mathbb{Z}}$$

the projective limit of the n-th roots of unity. When $k = \mathbb{C}$ we have $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$, and comparing with Example 1.3 we confirm that

$$\pi_1^{\mathrm{et}}(\mathbb{G}_{m/\mathbb{C}}; 1) = \pi_1\widehat{(\mathbb{C}^\times; 1)}$$

as it was stated at the outset. Moreover, we have:

$$\pi_1^{\mathrm{et}}(\mathbb{G}_{m/k}; 1, x) = \widetilde{f}^{-1}(x)$$

which is the projective limit of n-th roots of x.

Example 1.20. Suppose $\mathrm{char}(k) = 0$ and let $(A/k; o)$ be an abelian variety of dimension d. The maps $[n] : A \to A$ form a compatible system $\widetilde{A}_o = \{([n] : A \to A; o)\}_n$ in the sense of Theorem 1.17. Therefore:

$$\pi_1^{\mathrm{et}}(A; o) = \widetilde{f}^{-1}(o) = T(A) \simeq \widehat{\mathbb{Z}}^{2d}$$

the full Tate module of A (compare with Example 1.4). Moreover,

$$\pi_1^{\text{et}}(A; o, x) = \widetilde{f}^{-1}(x)$$

is a compatible system of n-division points of x.

1.3 The homotopy exact sequence

Suppose now we allow k to be *any* field, not necesserily algebraically closed. Then Definition 1.11 still makes sense, only in the definition of étale we require the various determinants to be non-zero possibly in finite field extensions of k.

Example 1.21. When $X = \text{Spec}(k)$ the theory is already highly non-trivial. Finite étale covers correspond to *finite étale aglebras* A over k, i.e. products of finite separable field extensions of k. We can then try and construct a 'universal covering space' $\widetilde{X}$ as in Theorem 1.17. Now the connected covers of X correspond to finite separable field extensions of k, and the Galois covers correspond to Galois extensions. The role of the base point is played by the scheme-theoretic point $\text{Spec}(\bar{k}) \to X$, for an algebraic closure $k \subset \bar{k}$. We form then a compatible system $\widetilde{X}_{\bar{k}} = \{L_\alpha, \text{Spec}(\bar{k}) \to \text{Spec}(L_\alpha)\}_\alpha$ in the sense of Theorem 1.17. Now for any Galois extension L_α/k the set $\text{Spec}(L_\alpha \otimes \bar{k})$, which is the fiber above $\text{Spec}(\bar{k}) \to X$, is a finite set of points indexed by the embeddings $L_\alpha \hookrightarrow \bar{k}$, which are all contained inside k_s. We deduce that:

$$\pi_1^{\text{et}}(X; \text{Spec}\,\bar{k} \to X) = \text{Gal}(k_s/k).$$

Therefore the theory of the étale fundamental group for $\text{Spec}(k)$ coincides with Galois theory over k.

As Example 1.21 shows, when k is not algebraically closed a new type of étale covers appear, which we may call of *arithmetic type*, as opposed to the ones of *geometric type* of the previous section. For example, if L/k is a finite separable field extension and X/k is a variety, then $X \times_k \text{Spec}(L)$ is a finite étale cover of X/k. Now if we choose a *geometric point*

$$\bar{b} : \text{Spec}(\bar{k}) \longrightarrow X$$

then it should be clear by now how to define the étale fundamental group $\pi_1^{\text{et}}(X; \bar{b})$. The following theorem of Grothendieck essentially shows that $\pi_1^{\text{et}}(X; \bar{b})$ can be computed by its geometric and arithmetic covers.

Theorem 1.22 (Homotopy exact sequence). *Let X/k be a geometrically irreducible variety over a field k. Fix an algebraic closure $\bar{k}$ and a separable closure $k_s \subset k$. Let $\bar{b} : \text{Spec}(\bar{k}) \to X$*

be a geometric point and let $\bar{X} := X \times_k \mathrm{Spec}(k_s)$*. Then the sequence of profinite groups*

$$1 \longrightarrow \pi_1^{\mathrm{et}}(\bar{X};\bar{b}) \longrightarrow \pi_1^{\mathrm{et}}(X;\bar{b}) \longrightarrow \mathrm{Gal}(k_s/k) \longrightarrow 1 \tag{13}$$

induced by the maps $\bar{X} \to X \to \mathrm{Spec}(k)$*, is exact.*

Proof. [Sza09] Proposition 5.6.1. □

We are now in a position to state Grothendieck's **section conjecture**, which was our main motivation for the study of the étale fundamental group. Note in fact that with the hypotheses of Theorem 1.22, any k-rational point

$$x : \mathrm{Spec}(k) \longrightarrow X$$

of X gives rise by functoriality to a map σ_x:

$$\mathrm{Gal}(k_s/k) \simeq \pi_1^{\mathrm{et}}(\mathrm{Spec}(k);\bar{k}) \xrightarrow{\sigma_x} \pi_1^{\mathrm{et}}(X;\bar{x})$$

where $\bar{x}$ is the geometric point obtained by base change to $\bar{k}$. If we compose with a choice of isomorphism:

$$\lambda : \pi_1^{\mathrm{et}}(X;\bar{x}) \longrightarrow \pi_1^{\mathrm{et}}(X;\bar{b})$$

then the composition:

$$\begin{array}{ccc} \pi_1^{\mathrm{et}}(X;\bar{b}) & \longrightarrow & \mathrm{Gal}(k_s/k) \\ {\scriptstyle\lambda}\nwarrow & & \swarrow{\scriptstyle\sigma_x} \\ & \pi_1^{\mathrm{et}}(X;\bar{x}) & \end{array}$$

is a section of the exact sequence (13). But the isomorphisms $\lambda : \pi_1^{\mathrm{et}}(X;\bar{x}) \to \pi_1^{\mathrm{et}}(X;\bar{b})$ are in bijection with the elements of

$$\pi_1^{\mathrm{et}}(X;\bar{b},\bar{x}) = \mathrm{Isom}(F_{\bar{b}}^{\mathrm{et}}, F_{\bar{x}}^{\mathrm{et}})$$

and any two isomorphisms λ are conjugate under $\pi_1^{\mathrm{et}}(X;\bar{b})$ (just as in the topological case). We therefore have a map:

$$X(k) \longrightarrow [\pi_1^{\mathrm{et}}(X;\bar{b},\bar{x})] = \{\text{ conjugacy classes of sections of (13)}\}$$

or, in the language of torsors, a map:

$$\begin{aligned} X(k) &\longrightarrow H^1(\mathrm{Gal}(k_s/k), \pi_1^{\mathrm{et}}(X; \bar{b})) \\ x &\longrightarrow [\pi_1^{\mathrm{et}}(X; \bar{b}, \bar{x})]. \end{aligned}$$

which is exactly the map (see (5)) that we set out to define .

Example 1.23. Let $(E; o)$ be an elliptic curve over $\mathbb{Q}$ and let $x \in E(\mathbb{Q})$ be a rational point. The associated torsor $\pi_1^{\mathrm{et}}(X; \bar{o}, \bar{x})$ is never trivial in $H^1(G_{\mathbb{Q}}, \pi_1^{\mathrm{et}}(E; \bar{o}))$. In fact, any $G_{\mathbb{Q}}$-invariant isomorphism

$$\pi_1^{\mathrm{et}}(E; \bar{o}) \xrightarrow{\simeq} \pi_1^{\mathrm{et}}(X; \bar{o}, \bar{x})$$

gives $G_{\mathbb{Q}}$-equivariant isomorphisms

$$E[n] \xrightarrow{\simeq} E[n]_x := \{P \in E(\overline{\mathbb{Q}}) : nP = x\}$$

for any $n \geq 1$, by Example 1.20. Since $o \in E[n](\mathbb{Q})$ for any n, this means that we can find a rational point $P_n \in E[n]_x(\mathbb{Q})$ for any n. Now a theorem of Mordell ([Sil85] III.6.4) shows that there exists a 'height function' $h : E(\mathbb{Q}) \to \mathbb{R}_{\geq 0}$ such that:

$$h(x) = h([n]P_n) \geq n^2 h(P_n) + C_E$$

where C_E is a constant depending only on E. As $n \to \infty$, this would require $h(P_n)$ to become arbitrarily small. But there are only finitely many points in $E(\mathbb{Q})$ of bounded height.

The main open question about the étale period map (5) is the following:

Conjecture 1.24 (Grothendieck's section conjecture). *Let X/k be a geometrically irreducible curve of genus ≥ 2 over a field k finitely generated over $\mathbb{Q}$. Then* (5) *is a bijection.*

The injectivity of (5), which seems plausible in light of Example 1.23, has been verified by Grothendieck. The surjectivity, however, remains an open question.

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