McGill University December 2012
Faculty of Science Final Examination

Abstract Algebra

Math 235

December, 2012

Time: 9:00 am - 12:00 pm

Examiner: Prof. Henri Darmon Associate Examiner: Prof. Eyal Goren

INSTRUCTIONS

1. Please answer questions in the exam booklet provided.
2. This exam consists of 10 questions. Each question is worth 10 points.
3. This a CLOSED BOOK exam. No notes, textbooks or crib sheets are permitted.
4. Calculators, cell phones, smart phones, tablets and laptops are not permitted.
5. Please write your name clearly on the examination booklet.

This exam comprises the cover page and two page of questions.

1. Compute the reduced residue modulo 13 (i.e., the unique integer r with $0 \leq r \leq 12$) of the integer

$$3^{1000000000000000000000000000000000000001}.$$

2. Give an example of a ring R and an ideal I in it which is *not* principal. Describe the quotient R/I for your choice.

3. Compute the gcd of the polynomials

$$x^p - x, \quad x^3 - 1,$$

in the ring $\mathbf{Z}/p\mathbf{Z}[x]$, when
a) p is a prime of the form $1 + 3m$;
b) p is a prime of the form $2 + 3m$.

4. Let F be a field and let $G = F^{\times}$ be its multiplicative group, i.e., its set of non-zero elements under the group operation. For all $t \geq 1$, show that G has at most t elements of order t.

5. State Fermat's little theorem, and state Lagrange's Theorem in group theory. Explain how the latter can be used to prove the former.

6. Show that the element $1+\sqrt{-11}$ is irreducible in the ring $\mathbf{Z}[\sqrt{-11}]$, and that the ideal generated by 3 and $1+\sqrt{-11}$ is not principal. What more familiar ring is the quotient $\mathbf{Z}[\sqrt{-11}]/(3,1+\sqrt{-11})$ isomorphic to?

7. Show that the group $\mathbf{GL}_2(\mathbf{Z}/2\mathbf{Z})$ of invertible 2×2 matrices with entries in $\mathbf{Z}/2\mathbf{Z}$ has cardinality 6, and that it is isomorphic to the symmetric group S_3 on three letters.

8. Let H be a subgroup of a group G and suppose that H has *index two* in G, i.e., that there are precisely two elements in G/H. Show that H is a normal subgroup of G.

9. Let H be a subgroup of a group G, and let N be a normal subgroup of G. Show that $H\cap N$ is a normal subgroup of H.

10. Recall that a group G is said to be *simple* if it has no non-trivial normal subgroups other than G and $\{1\}$. Show that any non-trivial homomorphism $f: G \longrightarrow G'$ (i.e., for which $f(G)\neq\{1\}$) is necessarily injective if G is a simple group.