189-235A: Algebra 1

Midterm Exam

Wednesday, October 25

Each of the four questions below is worth 25 points. No calculators or outside materials are allowed during the exam.

1. Let i be the complex number satisfying $i^2 = -1$. Compute $(1+i)^{100}$.

2. Let a and b be non-zero integers, and let L be the set of all *strictly positive* linear combinations of a and b:

$$L = \{ra + sb, \text{ with } r, s \in \mathbf{Z} \text{ and } ra + sb > 0\}.$$

(a) Show that the smallest element of L divides a and b.
(b) Show that this smallest element is the gcd of a and b.

3. Compute the reduced residue modulo N (i.e., the unique integer $0 \le x \le N-1$ with $x \equiv a \pmod{N}$) of the integer

$$a = 7^{13198459348751983475867345892342398209234983465234531}$$

for the following values of N.
(a) $N = 11$;
(b) $N = 5$;
(c) $N = 55$.

4. Solve the following congruence equations (making sure you list all the distinct solutions in $\mathbf{Z}/N\mathbf{Z}$).
(a) $5x = 2 \pmod{11}$.
(b) $10x = 4 \pmod{22}$.
(c) $10x = 3 \pmod{22}$.