189-235A: Basic Algebra I

Assignment 1

Due: Monday, September 24

1. Use Cardano's formula to solve the following cubic equations. In each case say how many real solutions there are and list all such solutions when there are more than one. (You are advised to use a calculator to check that the expressions you've written down are indeed solutions to the equation at hand.)
a. $x^3 + 3x + 1$
b. $x^3 - 3x + 1$. (In this case, give a closed form expression for the solution(s) of the equation, in terms of $\cos(2\pi/9)$ and $\sin(2\pi/9)$.)

2. Let S and T be the sets $\{a, b, c\}$and $\{x, y\}$ respectively.
a. How many functions are there from S to T?
b. How many injective functions?
b. How many surjective functions?

3. Let X be a set, and let $\mathcal{F}(X)$ be the set of all functions from X to itself. This set is equipped with a natural binary operation $(f, g) \mapsto fg$, given by the composition of functions.
a. Show that $f(gh) = (fg)h$ for all f, g, h in $\mathcal{F}(X)$. (In other words, the operation of composition of funcions is *associative*.)
b. Show, by providing an example, that fg need not be equal to gf, i.e., that composition of functions *need not be commutative*.

4. Show (*without resorting to a calculator or computer*!) that the complex number $(1 + \sqrt{3}i)^{111}$ is an integer, and write it down in factored form.

5. Show using induction that for all $n \geq 1$,

$$1^3 + \cdots + n^3 = (1 + \cdots + n)^2.$$

6. Using the Euclidean algorithm compute the gcd of 910091 and 3619. Show the steps in your calculation.

7. Using induction (or otherwise) show that 7 divides $8^n - 1$ for all $n \geq 0$. Use induction to show that 49 divides $8^n - 7n - 1$ for all $n \geq 0$.

8. Using induction, show that the addition law in $\mathbf{N}$ is associative directly from the axioms defining addition in $\mathbf{N}$.

9. Exercise (17), page 32 of the on-line notes.

10. Exercise (19), page 33 of the on-line notes.