189-726B: Modular Forms II
Assignment 5

Due: Wednesday, February 13

1. Fill in the details of the argument explained in class to show that the p-adic L-function $L_p^{\pm}(f,s)$ can be expressed as a convergent infinite sum:

$$L_p^{\pm}(f,s) = \sum_{j=0}^{\infty} a_j \binom{s-1}{j},$$

and give an explicit formula for the coefficients a_j that occur in this expansion, in terms of the moments of $\mu_f^{\pm}$ that were defined in class.

2. Give an explicit formula for the first derivative of $L_p^{\pm}(f,s)$ at $s=1$, in terms of these moments.

3. Suppose that N is a prime and that p is a primitive root mod N. Show that, if f is a weight 2 newform on $\Gamma_0(N)$, then the $L_p^{\pm}(f,s)$ do not vanish identically on the domain $(\mathbf{Z}/(p-1)\mathbf{Z}) \times \mathbf{Z}_p$.

4. What if the assumption on N is relaxed? (I think this problem is challenging. I remember proving some results in this direction, but I can't remember how far I got, or what my proof was. So the idea here is really to see how far you can go!)

Remark. Proving that $L_p^{\pm}(f,s)$ do not vanish identically on $(\mathbf{Z}/(p-1)\mathbf{Z}) \times \mathbf{Z}_p$ is (I think) a tractable problem. It is still not known how to show that $L_p(f,s)$ does not vanish identically on $\{1\} \times \mathbf{Z}_p$. It is widely believed that this is true. A proof would make an excellent PhD thesis!