Week 3, lecture 8: More on Modular Forms

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Recall

We are studying modular forms with respect to a congruence group Γ, i.e., a subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ such that $\Gamma \supseteq \Gamma(N)$ for some integer N.

Recall our notations:

$$\begin{array}{cc} \text{modular forms} & M_k(\Gamma) \\ & \cup \\ \text{cusp forms} & S_k(\Gamma) \end{array}$$

More on Modular Forms

Claim 1. (to be justified) *Cusp forms (of an appropriate type) play the same role for 2-dimensional representations of $G_{\mathbb{Q}}$ as do Dirichlet characters for 1-dimensional representations of $G_{\mathbb{Q}}$.*

From now on, we assume

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma.$$

This implies that, for any $f \in S_k(\Gamma)$,

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}.$$

We can then define the L-function attached to f by

$$L(f, s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

Question 1. *Where does this sum converge?*

In order to answer this question, we will introduce some basic results about Möbius transformations and modular forms. In what follows, we assume $f \in S_k(\Gamma)$.

Lemma 1. $y\left(\frac{az+b}{cz+d}\right) = \frac{y(z)}{(cz+d)(c\bar{z}+d)}$ *for all* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$.

Proof.
$$y\left(\frac{az+b}{cz+d}\right)=\frac{1}{2i}\left[\frac{az+b}{cz+d}-\overline{\left(\frac{az+b}{cz+d}\right)}\right]=\frac{1}{2i}\left[\frac{(ad-bc)(z-\overline{z})}{(cz+d)(c\overline{z}+d)}\right]=\frac{y(z)}{(cz+d)(c\overline{z}+d)}.$$

□

COROLLARY 1. *$y^k f(z)\overline{f(z)}$ is invariant under Γ. Hence, so is $y^{k/2}|f(z)|$.*

LEMMA 2. *There exists a constant C_f (depending on f) such that*
$$|f(x+iy)|<C_f y^{-k/2}$$
for all $z=x+iy\in\mathcal{H}$.

Proof. Since f is a cusp form, the values of $y^{k/2}|f(z)|$ are uniformly bounded in a neighborhood of all the cusps. Hence, there exists C_f such that
$$y^{k/2}|f(z)|<C_f.$$

□

PROPOSITION 1. *There exists a constant C_f' such that*
$$|a_n|<C_f' n^{k/2}.$$

Proof. Let $z=x+iy\in\mathcal{H}$. Then
$$f(z)=\sum_{n=1}^{\infty}a_n e^{2\pi in(x+iy)}=\sum_{n=1}^{\infty}a_n e^{-2\pi ny}e^{2\pi inx}.$$

By Fourier inversion,
$$a_n e^{-2\pi ny}=\int_0^1 f(x+iy)e^{-2\pi inx}dx.$$

So
$$|a_n|e^{-2\pi ny}\leq\int_0^1|f(x+iy)|dx\leq C_f y^{-k/2}.$$

And this implies that
$$|a_n|\leq C_f y^{-k/2}e^{2\pi ny}.$$

Notice that for each $y>0$ we have a bound for $|a_n|$. Choosing $y=\frac{k}{4\pi n}$ will minimize the function on the right hand side and we obtain
$$|a_n|\leq C_f' n^{k/2},$$
where $C_f'=C_f k^{-k/2}(4\pi)^{k/2}e^{k/2}$. □

Corollary 2. *$L(f,s)$ converges absolutely for $Re(s) > 1 + k/2$.*

Definition 1. The *Mellin transform* of f is

$$M(f)(s) = \int_0^\infty f(it)t^s \frac{dt}{t}.$$

Proposition 2. *$M(f)(s) = (2\pi)^{-s}\Gamma(s)L(f,s)$.*

Proof. The proof is similar to the proof of proposition 1 in lectures 5 and 6. □

Remark 1. If $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \Gamma$, then $M_k(\Gamma) = \{0\}$ when k is odd.

Theorem 2. *Assume $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$. Then $L(f,s)$ extends to an analytic function on $\mathbb{C}$ and, if*

$$\Lambda(f,s) := (2\pi)^{-s}\Gamma(s)L(f,s),$$

then

$$\Lambda(f,s) = (-1)^{k/2}\Lambda(f,k-s).$$

Proof.

$$\Lambda(f,s) = M(f)(s) = \int_0^\infty f(it)t^s \frac{dt}{t} = \int_0^1 f(it)t^s \frac{dt}{t} + \int_1^\infty f(it)t^s \frac{dt}{t}.$$

Applying the change of variables $t \leftrightarrow 1/t$ for the first integral gives us

$$\int_1^\infty f(i/t)t^{-s} \frac{dt}{t}.$$

Now, since

$$f(i/t) = f(-1/it) = (it)^k f(it),$$

the first integral becomes

$$i^k \int_1^\infty f(t)t^{k-s} \frac{dt}{t} = (-1)^{k/2} \int_1^\infty f(t)t^{k-s} \frac{dt}{t}.$$

Applying the same change of variables for the second integral gives us

$$(-1)^{k/2} \int_0^1 f(t)t^{k-s} \frac{dt}{t}.$$

So that

$$\Lambda(f,s) = (-1)^{k/2} \left(\int_1^\infty f(it)t^s \frac{dt}{t} + \int_0^1 f(it)t^s \frac{dt}{t} \right) = (-1)^{k/2}\Lambda(f,k-s)$$

□

REMARK 2. If $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ then the sign in the functional equation for $L(f,s)$ is

$$\begin{cases} 1, & \text{if } k \equiv 0 \pmod 4 \\ -1, & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

In particular, if $k \equiv 2 \pmod 4$, then $L(f, k/2) = 0$.

QUESTION 2. *Where are the zeros of $L(f,s)$ on $Re(s) > 1 + k/2$?*

QUESTIONS 3. *(i) The proof of analytic continuation works well when $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \Gamma$. What about other Γ (e.g. $\Gamma_0(N)$, $\Gamma_1(N)$)?*

(ii) When does $L(f,s)$ admit an Euler product factorization? And if so, what does it look like?

The key tool for these questions is

$$\boxed{\text{Hecke operators}}.$$