Math 726: L-functions and modular forms — Fall 2011

Week 2, lectures 5 and 6: The functional equation for Dirichlet L-functions. Remarks on Artin L-functions attached to higher-dimensional representations.

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Recall

Last lecture we explained L-functions attached to one dimensional representations of the absolute Galois group of $\mathbb{Q}$, $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. By Class Field Theory, we showed that $L(\rho, s) = L(\chi, s)$ for some Dirichlet character $\chi : (\mathbb{Z}/q\mathbb{Z})^{\times} \longrightarrow \mathbb{C}^{\times}$ for some integer q, called the conductor of χ. Throughout these lectures, we will assume that q is prime. As an exercise, think about the case of general conductor q.

Functional Equation of $L(\chi, s)$

For the case of an even characater, i.e. $\chi(-1) = 1$, we introduced

$$\Lambda(\chi, s) = \pi^{-s/2} q^{s/2} \Gamma\left(\frac{s}{2}\right) L(\chi, s) = \int_0^{\infty} \omega(t, \chi) t^{s/2} \frac{dt}{t} = M(\omega(t, \chi), \frac{s}{2}),$$

where $\omega(t, \chi) = \sum_{n=1}^{\infty} \chi(n) e^{-\pi n^2 t/q} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \chi(n) e^{-\pi n^2 t/q} = \frac{1}{2}\theta(t, \chi)$.

As with the Riemann ζ function, using the Poisson Summation Formula we will derive a functional equation for $\theta(t, \chi)$ of the form

$$\theta\left(\frac{1}{t}, \chi\right) = *\sqrt{t}\theta(t, \overline{\chi}),$$

where $*$ is a factor we will determine soon.

Last time we derived the equation

$$\sum_{n \in \mathbb{Z}} e^{-\pi(a+qn)^2/qt} = \sqrt{\frac{t}{q}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t/q} \psi_a(n), \tag{1}$$

where $\psi_a : \mathbb{Z}/q\mathbb{Z} \longrightarrow \mathbb{C}^{\times}$ is the additive character given by $\psi_a(n) = e^{2\pi i a n/q}$. Now, to relate the right hand side to $\theta(t, \chi)$ it is necessary to replace the additive character ψ_a by by the multiplicative character χ, or, more precisely, to express the latter as a linear combination of the characters ψ_a for some values of a. In order to do this we will do Fourier Analysis on the finite group $\mathbb{Z}/q\mathbb{Z}$.

Let $L^2(\mathbb{Z}/q\mathbb{Z},\mathbb{C}) = \{\mathbb{C}\text{-valued functions on } \mathbb{Z}/q\mathbb{Z}\}$ be the Hilbert space equipped with the inner product

$$\langle f, g\rangle = \frac{1}{q}\sum_{a\in\mathbb{Z}/q\mathbb{Z}} f(a)\overline{g(a)}.$$

The characters $\psi_0,\ldots,\psi_{q-1}$ are an orthonormal basis of $L^2(\mathbb{Z}/q\mathbb{Z},\mathbb{C})$ and in particular, for all $f\in L^2(\mathbb{Z}/q\mathbb{Z},\mathbb{C})$ we have $f=\sum_{j=0}^{q-1}\langle f,\psi_j\rangle\psi_j$. We are mainly interested in $f=\chi$, in which case we have the following properties:

(a) $\langle \chi,\psi_0\rangle = 0$ (because $\chi\neq 1$).

(b) $\langle \chi,\psi_1\rangle = \frac{1}{q}\sum_{j=0}^{q-1}\chi(j)e^{-2\pi ij/q} =: \frac{\tau(\chi)}{q}$. $\tau(\chi)$ is called the Gauss sum attached to χ. Some properties of the Gauss sums, which are left as an exercise are

- $\tau(\overline{\chi}) = \chi(-1)\overline{\tau(\chi)}$.
- $\tau(\chi)\overline{\tau(\chi)} = q$, i.e., $\|\tau(\chi)\| = \sqrt{q}$.

(c) If $a\in(\mathbb{Z}/q\mathbb{Z})^\times$, then $\exists a'\in(\mathbb{Z}/q\mathbb{Z})^\times$ such that $aa'\equiv 1 \pmod q$. As j runs from 0 to $q-1$, so does $a'j$ so

$$\begin{aligned}\langle\chi,\psi_a\rangle &= \frac{1}{q}\sum_{j=0}^{q-1}\chi(j)e^{-2\pi iaj/q} = \frac{1}{q}\sum_{j=0}^{q-1}\chi(a'j)e^{-2\pi iaa'j/q}\\ &= \frac{1}{q}\sum_{j=0}^{q-1}\chi(a'j)e^{-2\pi ij/q} = \overline{\chi}(a)\cdot\frac{1}{q}\sum_{j=0}^{q-1}\chi(j)e^{-2\pi ij/q}\\ &= \overline{\chi}(a)\frac{\tau(\chi)}{q}.\end{aligned}$$

From the previous properties we deduce that $\chi(n) = \frac{\tau(\chi)}{q}\sum_{a=0}^{q-1}\overline{\chi}(a)\psi_a(n)$. We multiply (1) by $\frac{\tau(\chi)}{q}\cdot\overline{\chi}(a)$ and we sum over $a=0,\ldots,q-1$ to obtain

$$\frac{\tau(\chi)}{q}\sum_{a=0}^{q-1}\overline{\chi}(a)\sum_{n\in\mathbb{Z}}e^{-\pi(a+qn)^2/qt} = \sqrt{\frac{t}{q}}\sum_{n\in\mathbb{Z}}e^{-\pi n^2t/q}\chi(n).$$

The LHS can be written as

$$\frac{\tau(\chi)}{q}\sum_{n\in\mathbb{Z}}\sum_{a=0}^{q-1}\overline{\chi}(a+qn)e^{-\pi(a+qn)^2/qt},$$

because $\overline{\chi}$ is q-periodic, but as a runs from 0 to $q-1$ and n runs over all integers, $a+qn$ runs over all integers exactly once, so the LHS simplifies to

$$\frac{\tau(\chi)}{q}\sum_{n\in\mathbb{Z}}\overline{\chi}(n)e^{-\pi n^2/qt}$$

and thus we obtain

$$\frac{\tau(\chi)}{q}\theta\left(\frac{1}{t},\overline{\chi}\right)=\sqrt{\frac{t}{q}}\theta(t,\chi),$$

which yields

$$\boxed{\theta\left(\frac{1}{t},\overline{\chi}\right)=\left(\frac{\sqrt{q}}{\tau(\chi)}\right)\cdot\sqrt{t}\theta(t,\chi)}$$

It is now easy to write down the functional equation for $\Lambda(\chi,s)$ and show that it extends analytically to the whole complex plane. We will apply exactly the same techniques we applied for the Riemann ζ-function; namely, we will use its expression as a Mellin transform, we will split the integral into two parts, one of them with nice convergence everywhere and in the other one we apply the substitution $t\mapsto 1/t$, where we can use the functional equation we just derived for $\theta(t,\chi)$ as follows:

$$\begin{aligned}2\Lambda(\chi,s)&=\int_0^\infty\theta(t,\chi)t^{s/2}\frac{dt}{t}=\int_0^1\theta(t,\chi)t^{s/2}\frac{dt}{t}+\int_1^\infty\theta(t,\chi)t^{s/2}\frac{dt}{t}\\&=\int_1^\infty\theta\left(\frac{1}{t},\chi\right)t^{-s/2}\frac{dt}{t}+\int_1^\infty\theta(t,\chi)t^{s/2}\frac{dt}{t}\\&=\frac{\sqrt{q}}{\tau(\overline{\chi})}\int_1^\infty\sqrt{t}\theta(t,\overline{\chi})t^{-s/2}\frac{dt}{t}+\int_1^\infty\theta(t,\chi)t^{s/2}\frac{dt}{t}\\&=\frac{\tau(\chi)}{\sqrt{q}}\int_1^\infty\theta(t,\overline{\chi})t^{\frac{1-s}{2}}\frac{dt}{t}+\int_1^\infty\theta(t,\chi)t^{s/2}\frac{dt}{t},\end{aligned}$$

which clearly is entire, so $\Lambda(\chi,s)$ extends analytically to the whole complex plane. Furthermore, we can see the relation

$$\boxed{\Lambda(\chi,s)=\left(\frac{\tau(\chi)}{\sqrt{q}}\right)\cdot\Lambda(\overline{\chi},1-s).}$$

Remarks:

- $\Lambda(\chi, s)$ is everywhere holomorphic (no poles at $s = 0$ and/or $s = 1$).

- The functional equations relates $\Lambda(\chi, s)$ to $\Lambda(\overline{\chi}, 1 - s)$ and involves a "root number" $\tau(\chi)/\sqrt{q}$.

- Since $L(\chi, s)$ admits an Euler product expansion for $\Re[s] > 1$, it does not vanish in said half-plane. The analyticity of $\Lambda(\chi, s)$ and the poles of $\Gamma(s)$ at the nonpositive integers yield zeroes of $L(\chi, s)$ at the nonpositive even integers. These are the so-called *trivial zeroes* of the L-function. The functional equation shows that there are no more zeroes in the half-plane $\Re[s] < 0$ and it is conjectured that all the zeroes in the critical strip (i.e., $0 \leq \Re[s] \leq 1$) lie on the line $\Re[s] = \frac{1}{2}$. This is the Generalized Riemann Hypothesis.

- Dirichlet proved that $L(\chi, 1) \neq 0$, which is a key ingredient in his proof of the theorem with his name on primes in arithmetic sequences.

Now we'll deal with odd characters, i.e., $\chi(-1) = -1$. Using exactly the same ideas as for even characters is useless because $\theta(t, \chi)$ is 0, unless we modify the definition of $\omega(t, \chi)$, so we will multiply by another function, or quasi-character, to make it an even character, namely, instead of $\chi(n)$ we will write $n\chi(n)$. More concretely, the idea we will use is defining

$$\omega(t, \chi) = \sum_{n=1}^{\infty} n\chi(n) e^{-\pi n^2 t/q} = \frac{1}{2} \sum_{n \in \mathbb{Z}} n\chi(n) e^{-\pi n^2 t/q} = \frac{1}{2}\theta(t, \chi).$$

Proposition 1. $M\left(\omega, \frac{s}{2}\right) = q^{s/2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right) L(\chi, s-1).$

Proof. It's just a simple computation.

$$\begin{aligned}
M\left(\omega, \frac{s}{2}\right) &= \int_0^{\infty} \omega(t, \chi) t^{s/2} \frac{dt}{t} = \int_0^{\infty} \sum_{n=1}^{\infty} n\chi(n) e^{-\pi n^2 t/q} t^{s/2} \frac{dt}{t} \\
&= \sum_{n=1}^{\infty} n\chi(n) \left(\int_0^{\infty} e^{-\pi n^2 t/q} t^{s/2} \frac{dt}{t} \right) \\
&= \sum_{n=1}^{\infty} n\chi(n) \int_0^{\infty} e^{-u} q^{s/2} \pi^{-s/2} n^{-s} u^{s/2} \frac{du}{u} \\
&= \left(\sum_{n=1}^{\infty} \chi(n) n^{1-s} \right) \Gamma\left(\frac{s}{2}\right) \pi^{-s/2} q^{s/2}
\end{aligned}$$

□

Corollary 1. $M\left(\omega, \frac{s+1}{2}\right) = q^{\frac{s+1}{2}} \pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) L(\chi, s).$ $\square$

Since we are using a different $\theta(t, \chi)$, we can't expect the same functional equation to work. Instead, we will derive another one. Recall the identity obtained from the Poisson Summation Formula,

$$\sum_{n \in \mathbb{Z}} e^{-\pi(x+n)^2/t} = \sqrt{t} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} e^{2\pi i n x}.$$

Since we have nice convergence properties on both sides of the equality, we are allowed to take derivatives with respect to x and exchange the limit processes of summation and differentiation, yielding the equation

$$\sum_{n \in \mathbb{Z}} \frac{-2\pi(x+n)}{t} e^{-\pi(x+n)^2/t} = \sqrt{t} \sum_{n \in \mathbb{Z}} 2\pi i n e^{-\pi n^2 t} e^{2\pi i n x},$$

whence

$$i \sum_{n \in \mathbb{Z}} (x+n) e^{-\pi(x+n)^2/t} = t^{3/2} \sum_{n \in \mathbb{Z}} n e^{-\pi n^2 t} e^{2\pi i n x}.$$

Set $x = \frac{a}{q}$, $a \in \mathbb{Z}$ and replace t by $\frac{t}{q}$ to obtain

$$i \sum_{n \in \mathbb{Z}} (a+qn) e^{-\pi(a+qn)^2/tq} = \frac{t^{3/2}}{\sqrt{q}} \sum_{n \in \mathbb{Z}} n e^{-\pi n^2 t/q} \psi_a(n).$$

After multiplying by $\frac{\tau(\chi)}{q} \cdot \overline{\chi}(a)$ on both sides and summing over $a = 0, \ldots, q-1$ we find that

$$i \cdot \frac{\tau(\chi)}{q} \sum_{n \in \mathbb{Z}} n \overline{\chi}(n) e^{-\pi n^2/tq} = \frac{t^{3/2}}{\sqrt{q}} \sum_{n \in \mathbb{Z}} n \chi(n) e^{-\pi n^2 tq}$$

and it follows that

$$\theta\left(\frac{1}{t}, \overline{\chi}\right) = \left(\frac{\sqrt{q}}{i\tau(\chi)}\right) \cdot t^{3/2} \theta(t, \chi).$$

Now we get the following corollary.

Corollary 2. *Let χ be an odd Dirichlet character. The function*

$$\Lambda(\chi, s) = q^{\frac{s+1}{2}} \pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) L(\chi, s)$$

satisfies the functional equation

$$\Lambda(\chi, s) = \left(\frac{i\tau(\chi)}{\sqrt{q}}\right) \Lambda(\overline{\chi}, 1-s)$$

and is everywhere holomorphic.

Proof. Exercise. (Hint: Follow the walkthrough given for even characters.) $\square$

COROLLARY 3. *If χ is an odd Dirichlet character, $L(\chi, s)$ is an entire function with zeroes at all negative odd integers. Moreover, it vanishes nowhere else outside of the critical strip $0 \leq \Re[s] \leq 1$.* $\square$

Question: What about more general continuous $\rho : G_{\mathbb{Q}} \longrightarrow \mathbf{GL}_d(\mathbb{C}) \cong \mathrm{Aut}(V_\rho)$, $V_\rho \cong \mathbb{C}^d$,

$$L(\rho, s) = \prod_p \det(1 - \rho(\sigma_p)p^{-s} \circlearrowleft V_\rho^{I_p})^{-1}$$
$$= \sum_{n=1}^{\infty} a_n n^{-s} \qquad a_p = \mathrm{tr}(\sigma_p \circlearrowleft V_\rho),$$

where σ_p is the Frobenius element at p? What kind of patterns do the coefficients in this Dirichlet series satisfy?

Artin-Conjecture: Let ρ be any continuous non trivial irreducible representation of $G_{\mathbb{Q}}$. Then $L(\rho, s)$ extends to an analytic function on all of $\mathbb{C}$. Moreover, there is a prediction for what the functional equation looks like!! (Generalization of Class Field Theory to nonabelian representations.)

Status of the conjecture:

(a) It is known for $d = 2$, $\det \rho(-1) = -1$. This was finished around 2006 putting together the work of Hecke, Langlands-Tunnell, Serre-Deligne, Wiles, Taylor, Khare-Wintenberger. It is highly nontrivial.

(b) Very little is known for $d > 2$.

A much easier result is available.

THEOREM 1 (ARTIN). *The function $L(\rho, s)$ extends to a meromorphic function of $s \in \mathbb{C}$.*

Sketch of Proof. Observe that if ρ is a general representation, we can write $\rho = \bigoplus_{i=1}^{t} m_i \rho_i$, where $m_i \in \mathbb{Z}^{>0}$ and the ρ_i are irreducible. $L(\rho, s) = \prod_{i=1}^{t} L(\rho_i, s)^{m_i}$. $\square$

Key remark: Class Field Theory over a field K allows us to analyse the L-function $L_K(\rho, s)$ attached to $\rho : G_K := \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathbb{C}^\times$,

$$L_K(\rho, s) = \prod_{\wp \lhd \mathcal{O}_K} (1 - \rho(Frob_\wp)N(\wp)^{-s})^{-1} \quad (= 0 \text{ if } \rho(I_\wp) \neq 1).$$

Class Field Theory for K shows that $L_K(\rho, s) = L_K(\chi, s)$ where χ is a "Hecke character" of K.

THEOREM 2 (HECKE(TATE'S THESIS)). *$L_K(\rho, s)$ extends to a holomorphic function of s when $\rho \neq 1$ (with eventually a pole at $s = 1$).*

We have the examples $\zeta_K(s) = L(Ind_K^{\mathbb{Q}}\mathbf{1_K}, \mathbf{s})$ (assignment) and $L_K(\chi, s) = L(Ind_K^{\mathbb{Q}}\chi, s)$, where $Ind_K^{\mathbb{Q}}$ is a one dimensional representation of $G_{\mathbb{Q}}$. Then, Artin's conjecture is true for all ρ which are induced from abelian characters of K (K varying). More generally, if there exist $K_1, \ldots, K_t$ number fields and $\chi_1, \ldots, \chi_t$ Hecke characters of these fields such that $\rho = \bigoplus_{i=1}^{t} m_i Ind_{K_i}^{\mathbb{Q}}\chi_i$ with $m_i \geq 1$ then $L(\rho, s)$ satisfies Artin's conjecture.

THEOREM 3 (BRAUER'S THEOREM). *Let ρ be any representation of a finitely generated group G. Then, there exist*

- *subgroups $H_1, \ldots, H_t \leq G$*
- *charachters $\chi_j : H_j \longrightarrow \mathbb{C}^\times$*
- *(not necessarily positive) integers $m_1, \ldots, m_t$*

such that

$$\rho = \sum_{i=1}^{t} m_i Ind_{H_i}^{G}\chi_i$$

□

When applied to $\rho : \mathrm{Gal}(K/\mathbb{Q}) \longrightarrow \mathbf{GL}_d(\mathbb{C})$ we get $K_i = K^{H_i}$ and χ_i such that

$$L(\rho, s) = \prod_{i=1}^{t} L_{K_i}(\chi_i, s)^{m_i}.$$