# Week 3, lecture 9: Examples of Modular Forms

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## Eisenstein Series

Let $z \in \mathcal{H}$ and $k \in \mathbb{Z}_{>2}$ even.

We define

$$G_k(z) := \sum_{(m,n)\in\mathbb{Z}^2}{}' (mz+n)^{-k},$$

where the primed sum means we sum over all the pairs $(m, n)$ except $(0, 0)$.

FACT 1. *It is easy to see that this sum converges absolutely and uniformly on compact subsets of $\mathcal{H}$. So $G_k(z)$ is holomorphic on $\mathcal{H}$.*

Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbb{Z}^2 = \mathbb{Z}^2,$$

we obtain that

$$\begin{aligned} G_k\left(\tfrac{az+b}{cz+d}\right) &= \sum{}' \left[m\left(\tfrac{az+b}{cz+d}\right)+n\right]^{-k} = (cz+d)^k \sum{}' [m(az+b)+n(cz+d)]^{-k} \\ &= (cz+d)^k \sum{}' [(am+nc)z+(bn+dn)]^{-k} = (cz+d)^k G_k(z). \end{aligned}$$

We also define

$$E_k(z) := \frac{1}{2}\sum_{gcd(m,n)=1}{}' (mz+n)^{-k} = \frac{1}{\zeta(2k)} G_k(z) = 1 + \cdots$$

(the sum is multiplied by $1/2$ in order to ensure that the first coefficient in the Fourier expansion is 1).

THEOREM 1. *The Fourier expansion of $E_k$ is given by*

$$E_k(z) = 1 + \frac{(2\pi i)^k}{(k-1)!\zeta(k)} \sum_{n=1}^{\infty} \frac{n^{k-1}q^n}{1-q^n} = 1 + \frac{(2\pi i)^k}{(k-1)!\zeta(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

*where* $\sigma_{k-1}(n) = \sum_{d>0\ ,\ d|n} d^{k-1}$ *and* $q = e^{2\pi i z}$.

*Proof.* We start with the Euler's product expansion for $\sin(\pi z)$:

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Taking the logarithmic derivative on both sides results

$$\frac{\pi \cos(\pi z)}{\sin(\pi z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n}\right).$$

The left-hand side can be written as

$$\pi i \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} = \pi i \frac{e^{2\pi i z} + 1}{e^{2\pi i z} - 1} = \pi i \frac{q+1}{q-1} = -\pi i \left(1 + \frac{2q}{1-q}\right).$$

This implies that

$$\frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n}\right) = -\pi i - 2\pi i \sum_{n=1}^{\infty} q^n.$$

Applying $\left(\frac{d}{dz}\right)^{k-1}$ to both sides yields

$$(-1)^{k-1}(k-1)! \left[\frac{1}{z^k} + \sum_{n=1}^{\infty} \left(\frac{1}{(z+n)^k} + \frac{1}{(z-n)^k}\right)\right] = -(2\pi i)^k \sum_{n=1}^{\infty} n^{k-1} q^n.$$

Hence, since $k$ is even,

$$\sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q^n,$$

which in turn implies that

$$2 \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \frac{1}{(mz+n)^k} = 2 \frac{(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{k-1} q^{mn} = 2 \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

The terms with $m = 0$ contribute with

$$\sum_{n \in \mathbb{Z} - \{0\}} \frac{1}{n^k} = 2\zeta(k).$$

Therefore

$$G_k(z) = 2\zeta(k) + 2 \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

and

$$E_k(z) = 1 + \frac{(2\pi i)^k}{(k-1)!\zeta(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

□

EXERCISE 1. *Use the functional equation of the Riemann zeta function $\zeta$ to show that*

$$\frac{(2\pi i)^k}{(k-1)!\zeta(k)} = \frac{2}{\zeta(1-k)}$$

*when $k$ is even.*

EXAMPLE 2. We can now compute the first few Eisenstein series:

$E_4(q) = 1 + 240(q + 9q^2 + 28q^3 + 73q^4 + \cdots)$
$E_6(q) = 1 - 504(q + 33q^2 + 244q^3 + \cdots)$
$E_8(q) = 1 + 480(q + 129q^2 + \cdots)$
$E_6^2(q) = 1 - 1008q + \cdots$
$E_4(q)E_8(q) = 1 + 720q + \cdots$

## Ramanujan $\Delta$-function

The *Ramanujan $\Delta$-function* is the non-zero cusp form of weight 12 defined by

$$\Delta(q) := \frac{E_4(q)E_8(q) - E_6^2(q)}{1728} = q + \cdots .$$

PROPOSITION 1. *(1) If $k \le 2$ or $k$ is odd, then*

$$\dim_{\mathbb{C}} S_k(\mathrm{SL}_2(\mathbb{Z})) = 0.$$

*(2) Multiplication by $\Delta$ induces an isomorphism of $\mathbb{C}$-vector space*

$$M_k(\mathrm{SL}_2(\mathbb{Z})) \xrightarrow{\sim} S_{k+12}(\mathrm{SL}_2(\mathbb{Z})).$$

*(3) If $k \ge 4$, then*

$$\dim_{\mathbb{C}} M_k(\mathrm{SL}_2(\mathbb{Z})) = 1 + \dim_{\mathbb{C}} S_k(\mathrm{SL}_2(\mathbb{Z})).$$

*Proof.* Suppose $k = 0$. Then $M_0(\mathrm{SL}_2(\mathbb{Z}))$ is the space of $\mathrm{SL}_2(\mathbb{Z})$-invariant functions on $\mathcal{H}^*$ with rapid decay. These functions are bounded holomorphic functions on the compact Riemann surface $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}^*$. Hence

$$S_0(\mathrm{SL}_2(\mathbb{Z})) = \{0\} \text{ and } M_0(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}\,.$$

*(The proof will be continued in the next class.)* □