Math 726: L-functions and modular forms — Fall 2011

Week 2, lecture 7: Modular Forms and Congruence Subgroups

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Recall

Previously we dealt with one dimensional representations of $G_{\mathbb{Q}}$ and we established a correspondence, via Class Field Theory, between them and primitive Dirichlet characters. Also, we mentioned the correspondence between one dimensional representations of G_K (where K is a number field) and primitive Hecke characters, although we said much less about this.

Now we will deal with Artin representations of $G_{\mathbb{Q}}$ of dimension greater than 1, focusing on the two dimensional representations. What object would they be in correspondence with, analogous to the Dirichlet and Hecke characters in the one dimensional case? The Modular Forms! So, we are going to talk about Modular Forms.

More on the θ function.

For the functional equation of $\zeta(s)$ we used the function $\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}$, for which the relation $\theta\left(\frac{1}{t}\right) = \sqrt{t}\theta(t)$ holds. In all this analysis, $\theta : \mathbb{R}^{>0} \to \mathbb{R}$. It turns out to be better to consider it as a function on the right-half plane $\mathbb{C}^{\Re[s]} > 0$, $\theta : \mathbb{C}^{\Re[s]>0} \to \mathbb{C}$, so for convenience, we are going to rotate it to the Poincar upper-half plane $\mathcal{H} := \{z \in \mathbb{C} : \Im[z] > 0\}$ by defining the function $\tilde{\theta}(z) = \theta(-iz)$, so $\tilde{\theta}(z) = \displaystyle\sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}$.

Lemma 1. *For all $z \in \mathcal{H}$ the following relations hold:*

a) $\tilde{\theta}(z) = \tilde{\theta}(z+2)$

b) $\tilde{\theta}\left(\frac{-1}{z}\right) = e^{-2\pi i/8}\sqrt{z}\tilde{\theta}(z)$, *where the natural branch for the square root is chosen.*

Proof. *a)* is clear. For *b)*, we will check that it is true for the set $i\mathbb{R}^{>0}$, which has accumulation points. Since both sides of the equation are analytic functions on $\mathcal{H}$, if they agree on a set with accumulation points they will agree on the whole domain. Every $z \in i\mathbb{R}^{>0}$ can be expressed as iy with $y > 0$, so the relation reduces to $\theta\left(\dfrac{1}{y}\right) = \sqrt{y}\theta(y)$, which we know is true. $\square$

We can think of $z \mapsto z+2$ as $z \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}(z)$ and $z \mapsto \dfrac{-1}{z}$ as $z \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(z)$ acting as Mbius transformations. The two matrices generate a subgroup of the modular group $\mathbf{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}, ad-bc = 1 \right\}$. $\tilde{\theta}$ has nice transformation properties under this group action, which we will see later.

Subgroups of $\mathbf{SL}_2\mathbb{Z}$

The most important subgroups of $\mathbf{SL}_2(\mathbb{Z})$ have their own names. They are

$$\begin{aligned}
\Gamma_0(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\} \\
\Gamma_1(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\} \\
\Gamma(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}.
\end{aligned}$$

We have the chain of subgroups $\Gamma(N) \triangleleft \Gamma_1(N) \triangleleft \Gamma_0(N) \leq \mathbf{SL}_2(\mathbb{Z})$. Note that $\Gamma(N)$ is the kernel of the projection map $\mathbf{SL}_2(\mathbb{Z}) \to \mathbf{SL}_2(\mathbb{Z}/N\mathbb{Z})$, so $\Gamma(N) \triangleleft \mathbf{SL}_2(\mathbb{Z})$. We can identify the quotients as follows:

$$\begin{aligned}
\mathbf{SL}_2(\mathbb{Z})/\Gamma_0(N) &= \mathbb{P}_1(\mathbb{Z}/N\mathbb{Z}) \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto (a:c) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(\infty)
\end{aligned}$$

$$\begin{aligned}
\Gamma_0(N)/\Gamma_1(N) &= (\mathbb{Z}/N\mathbb{Z})^\times \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto d \quad (\text{or } a)
\end{aligned}$$

$$\begin{aligned}
\Gamma_1(N)/\Gamma(N) &= \mathbb{Z}/N\mathbb{Z} \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto b.
\end{aligned}$$

The first quotient is not a group, since $\Gamma_0(N)$ is, in general, not a normal subgroup of $\mathbf{SL}_2(\mathbb{Z})$. However, there is a left action of $\mathbf{SL}_2(\mathbb{Z})$ on the cosets which passes to $\mathbb{P}_1(\mathbb{Z}/N\mathbb{Z})$. The action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on $(A:C)$ results in $(aA+bC : cA+dC)$. The two morphisms $\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times$ and $\Gamma(1) \to \mathbb{Z}/NZ$ described earlier are well defined and they have kernels $\Gamma_1(N)$ and $\Gamma(N)$ respectively. Note that $ad \equiv 1 \pmod N$ because $ad - bc = 1$ and $N \mid c$. Also $(\mathbb{Z}/N\mathbb{Z})^\times$ is abelian, so the map $g \mapsto g^{-1}$ is an automorphism. This implies that the morphism $\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times$ can be defined as $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a$ as well if we want.

Modular Forms

Definition 1. A **modular form** of weight k, level N and character $\mathcal{E}$ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying

$$\boxed{f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z)\right) = f\left(\frac{az+b}{cz+d}\right) = \mathcal{E}(d)(cz+d)^k f(z)} \tag{1}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, together with suitable growth conditions at the cusps (which we will mention later).

Remark: We could have chosen to write $\mathcal{E}(a)$ instead. Everything would work the same as long as we are consistent with our choice.

Note that if $N = 1$, $\Gamma_0(N) = \mathbf{SL}_2(\mathbb{Z})$ and all characters are trivial, (1) reduces to

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z)\right) = (cz+d)^k f(z) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z}).$$

<u>More about these growth conditions at ∞.</u>

We know that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ so $f(z+1) = f(z)$, so we have a Fourier expansion for $f(z)$ as

$$f(z) = \sum_{n \in \mathbb{Z}} a_n q^n, \quad q = e^{2\pi i z}.$$

This is the so-called q-expansion of f.

Definition 2. We say that f is meromorphic (resp. holomorphic, resp. vanishes) if the q-expansion of $f(z)$ lies in $\mathbb{C}((q))$ (resp. $\mathbb{C}[[q]]$, resp. $q\mathbb{C}[[q]]$).

The values for a modular form of level 1 are predetermined by the values the function takes on at the region

$$\mathcal{F} := \left\{ z \in \mathbb{C} : -\frac{1}{2} \le \Re[z] < \frac{1}{2}, \|z\| > 1 \right\} \cup \left\{ z \in \mathbb{C} : -\frac{1}{2} \le \Re[z] \le 0, \|z\| = 1 \right\}.$$

A fundamental region for $\Gamma_1(N)$ is $\mathcal{F}_{\Gamma_1(N)} := \bigcup_{\gamma \in \Gamma_1(N) \backslash \mathbf{SL}_2(\mathbb{Z})} \gamma\mathcal{F}$, a finite union of hyperbolic triangles whose vertices lie in $\mathbb{P}_1(\mathbb{Q})$. This construction is not particular for $\Gamma_1(N)$; it works for all subgroups Γ such that there exists a positive integer n such that $\Gamma(N) \leq \Gamma \leq \mathbf{SL}_2(\mathbb{Z})$.
Weight k "Slash operators": If $\gamma \in \mathbf{SL}_2(\mathbb{Z})$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we define

$$f \mid_k \gamma(z) = (cz+d)^{-k} f(\gamma(z)).$$

Some properties of the slash operator:

1) If $f \mid_k \gamma = f$ for all $\gamma \in \Gamma_1(N)$, then f satisfies (1) for all $\gamma \in \Gamma_1(N)$.

2) $f \mid_k (\gamma_1\gamma_2) = (f \mid_k \gamma_1) \mid_k \gamma_2$.

3) If $f \mid_k \gamma = f$ for all $\gamma \in \Gamma$ and $\alpha \in \mathbf{SL}_2(\mathbb{Z})$, then $f \mid_k \alpha$ is invariant under $\alpha^{-1}\Gamma\alpha$.

In particular, if $\alpha(\infty) = r/s$, then $f \mid_k \alpha$ is invariant under some finite index subgroup of $\mathbf{SL}_2(\mathbb{Z})$. As a consequence, $\exists w \in \mathbb{Z}$ such that $f \mid_k \alpha(z) = f \mid_k \alpha(z+w)$. The minimal w with this property is called the **width** of the cusp r/s relative to Γ. Since the function is w-periodic, it has a Fourier expansion of the form

$$f \mid_k \alpha(z) = \sum_{n \in \mathbb{Z}} a_n(r/s) e^{2\pi i n z/w}.$$

Definition 3. The function f is holomorphic at r/s if $f \mid_k \alpha$ is holomorphic at ∞.

The suitable growth conditions at the cusps are precisely these, i.e., f is holomorphic at the cusps.
Notations:

a) $M_k(\Gamma_0(N), \mathcal{E})$ is the space of modular forms of weight k, level N and character $\mathcal{E}$ (holomorphic and holomorphic at the cusps).

b) $S_k(\Gamma_0(N), \mathcal{E})$ is the space of modular forms of weight k, level N and character $\mathcal{E}$ such that they vanish at all cusps (and are holomorphic.) Note that $S_k(\Gamma_0(N), \mathcal{E}) \subseteq M_k(\Gamma_0(N), \mathcal{E})$.

If $f \in M_k(\Gamma_0(N), \mathcal{E})$ and we write $f(z) = \sum_{n=0}^{\infty} a_n q^n$, we formally define the L-function attached to f to be

$$L(f,s) := \sum_{n=1}^{\infty} a_n n^{-s}.$$

Note the absence of the term a_0. This is a natural generalization of $L(\chi, s)$ as we will see. Next time we will show that the a_n have at most polynomial growth, $\|a_n\| \ll n^{k/2}$ (in fact, $\|a_n\| << n^{\frac{k-1}{2}}$, but this result is beyond the scope of this course).