Math 726: L-functions and modular forms — Fall 2011

Lecture 4: Dirichlet L-functions

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In the past two lectures we have studied the Riemann ζ function, which is the L-function attached to the trivial representation of $G_{\mathbb{Q}}$. Later we will study in more detail the zeroes and the critical values of ζ, but in this lecture we will move on to the 'next' natural case of L-functions, those attached to general 1-dimensional representations of $G_{\mathbb{Q}}$.

Consider then a continuous homomorphism:

$$\rho : G_{\mathbb{Q}} \longrightarrow \mathbb{C}^{\times}$$

where

$$G_{\mathbb{Q}} = \varprojlim_{[L:\mathbb{Q}]<\infty} \mathrm{Gal}(L/\mathbb{Q})$$

is given the Krull (profinite) topology. This topology has $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$, $[L:\mathbb{Q}] < \infty$ as a base of open subgroups. On the other hand the group $\mathbb{C}^{\times}$, endowed with the usual topology, does not have such a base and as a consequence we see that the representation ρ has to factor through a finite quotient:

$$\begin{tikzcd} G_{\mathbb{Q}} \arrow[rr, "\rho"] \arrow[dr] & & \mathbb{C}^{\times} \\ & \mathrm{Gal}(L_\rho/\mathbb{Q}) \arrow[ur, hook] & \end{tikzcd}$$

Note that from the injection $\mathrm{Gal}(L_\rho/\mathbb{Q}) \hookrightarrow \mathbb{C}^{\times}$ we deduce that $L_\rho/\mathbb{Q}$ is a (finite) **cyclic abelian** extension of $\mathbb{Q}$.

Recall from the first lecture that the L-function of ρ is defined as:

$$L(\rho, s) := \prod_p \frac{1}{1 - \rho(\mathrm{Frob}_p)|_{V^{I_p}} \cdot p^{-s}}. \tag{1}$$

Since V in this case is 1-dimensional, either $V^{I_p} = V$ (in which case we say that the representation is **unramified** at the prime p) or $V^{I_p} = 0$, in which case the p-factor in the L-function is equal to 1. Therefore we can rewrite (1) as:

$$L(\rho, s) := \prod_{p \text{ unramified}} \frac{1}{1 - \rho(\mathrm{Frob}_p) \cdot p^{-s}}.$$

Computation of the L-function then boils down to the following question:

Question 1. What is $\rho(\mathrm{Frob}_p)$ as a function of p?

For a general L-function, this could be a very complicated function. Thankfully in our case we can find the solution using class field theory. We first illustrate this computation with the simplest example, that of a quadratic extension.

Example 2. Consider a 1-dimensional representation:

$$\rho : G_{\mathbb{Q}} \longrightarrow \{\pm 1\}.$$

In this case L_ρ is a quadratic extension $L_\rho = \mathbb{Q}(\sqrt{D})$ where D is a square-free integer. The inertia group I_p is trivial if and only if $p \nmid 2D$ and for such unramified primes we have:

$$\rho(\mathrm{Frob}_p) = \begin{cases} 1 & \text{if } D \text{ is a square mod } p \\ -1 & \text{if } D \text{ is not a square mod } p. \end{cases}$$

From the **quadratic reciprocity law**, we deduce that $\rho(\mathrm{Frob}_p)$ only depends on the value of p modulo $4D$.

The basic examples of abelian extensions of $\mathbb{Q}$ are **cyclotomic fields**. For $n \geq 1$, let ζ_n be a primitive n-th root of unity and consider the field extension $\mathbb{Q}(\zeta_n)$. This is a Galois extension with cyclic abelian Galois group

$$(\mathbb{Z}/n\mathbb{Z})^\times \simeq \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$$
$$a \longmapsto (\zeta_n \mapsto \zeta_n^a)$$

where Frob_p sends ζ_n to ζ_n^p, i.e. under the above isomorphism we have:

$$p \longmapsto \mathrm{Frob}_p\,.$$

The following theorem asserts that these are essentially all the abelian extensions of $\mathbb{Q}$.

Theorem 3 (Kronecker-Weber Theorem, Class Field Theory for $\mathbb{Q}$). *Every abelian extension of $\mathbb{Q}$ is contained in a cyclotomic field $\mathbb{Q}(\zeta_n)$.*

By the Kronecker-Weber Theorem, our field extension L_ρ is contained in a cyclotomic extension. The smallest such extension controls ramification in L_ρ, since the only primes which ramify in $\mathbb{Q}(\zeta_n)$ are the ones dividing n. We therefore single out this extension:

Definition 4. The smallest $n \geq 1$ such that $L_\rho \subset \mathbb{Q}(\zeta_n)$ is called the **conductor** of L_ρ (or of ρ).

If n is the conductor of L_ρ, then thanks to Kronecker-Weber we obtain a multiplicative homomorphism $\chi : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^\times$ defined by the composition:

$$\begin{tikzcd} (\mathbb{Z}/n\mathbb{Z})^\times \arrow[r, "\simeq"] \arrow[rrr, bend right, "\chi"'] & \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \arrow[r] & \mathrm{Gal}(L_\rho/\mathbb{Q}) \arrow[r, "\rho"] & \mathbb{C}^\times \end{tikzcd}$$

We call χ a **Dirichlet character** of conductor n. Note that:

$$\rho(\mathrm{Frob}_p) = \chi(p)$$

and therefore:

$$L(\rho, s) = L(\chi, s) = \prod_p \left(1 - \chi(p)p^{-s}\right)^{-1} = \sum_{n=1}^{\infty} \chi(n)n^{-s}$$

where we used the convention:

$$\chi(p) = 0 \quad \text{if } \; p \mid n$$

(i.e. when p is ramified). We call this type of L-function a **Dirichlet L-function**.

We would now like to understand the analytic properties of $L(\chi, s)$. In order to simplify some of the arguments, we make the:

Simplifying assumption: The conductor $n = q$ is prime.

How do we go about studying $L(\chi, s)$? Informed by our treatment of the Riemann zeta function, the natural idea is to study the function:

$$\omega(t, \chi) = \sum_{n=1}^{\infty} \chi(n)e^{-\pi n^2 t/q}$$

which will play the role of $\omega(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 t}$ (the reason for the q-th root in the exponent will be apparent shortly). As a first step, we would like to write $L(\chi, s)$ as a Mellin transform of $\omega(t, \chi)$.

Proposition 5. *Let $\Lambda(\chi, s) := \pi^{-s/2} q^{s/2} \Gamma(s/2) L(\chi, s)$. Then*

$$\Lambda(\chi, s) = \int_0^\infty \omega(t, \chi) t^{s/2} \frac{dt}{t} = M(\omega(t, \chi))(s/2)$$

Proof. The proof is similar to the case $\chi = 1$. By definition of Mellin transform, we have:

$$M(\omega(t, \chi))(s) = \int_0^\infty \omega(t, \chi) t^s \frac{dt}{t} = \int_0^\infty \left(\sum_{n=1}^{\infty} \chi(n) e^{-\pi n^2 t/q} \right) t^s \frac{dt}{t}.$$

All the terms in the infinite series are of rapid decay, and therefore we can switch the order of integration:

$$\int_0^\infty \left(\sum_{n=1}^\infty \chi(n) e^{-\pi n^2 t/q} \right) t^s \frac{dt}{t} = \sum_{n=1}^\infty \int_0^\infty \chi(n) e^{-\pi n^2 t/q} \cdot t^s \frac{dt}{t}.$$

For each term in the series, we make the change of variables $u = \pi n^2 t/q$ to obtain:

$$\begin{aligned}
\sum_{n=1}^\infty \int_0^\infty \chi(n) e^{-\pi n^2 t/q} \cdot t^s \frac{dt}{t} &= \sum_{n=1}^\infty \int_0^\infty \chi(n) e^{-u} \pi^{-s} n^{-2s} q^s \cdot u^s \frac{du}{u} \\
&= \pi^{-s} \cdot q^s \cdot \left(\int_0^\infty e^{-u} u^s \frac{du}{u} \right) \cdot \left(\sum_{n=1}^\infty \chi(n) n^{-2s} \right) \\
&= \pi^{-s} q^s \Gamma(s) L(\chi, s).
\end{aligned}$$

□

What about the functional equation? Recall that we derived the functional equation for Λ from the functional equation for ω. This, in turn, was derived from the functional equation for θ, which was proved using Poisson summation. Since Poisson summation only applies for sums over $\mathbb{Z}$, we are prompted to define:

$$\theta(t, \chi) := \sum_{n \in \mathbb{Z}} \chi(n) e^{-\pi n^2 t/q}.$$

There are two important differences between the previous case of $\chi = 1$. In the first place, when χ is nontrivial then $\chi(0) = 0$ and therefore the term corresponding to $n = 0$ in the series defining $\theta(t, \chi)$ disappears. Secondly, the relation between χ and ω changes according to the value of χ at -1. Since $\chi(-1)^2 = 1$, there are only two cases:

$$\theta(t, \chi) = \begin{cases} 2\omega(t, \chi) & \text{if } \chi(-1) = 1 \\ 0 & \text{if } \chi(-1) = -1. \end{cases}$$

which will correspond to two different functional equations for $L(\chi, s)$. Accordingly, we make the following definition:

Definition 6. The Dirichlet character χ is **even** if $\chi(-1) = 1$ and it is **odd** if $\chi(-1) = -1$.

When χ is odd the theta function is zero and we have to find a different approach to the functional equation. Therefore we begin by analyzing the case χ is even:

Assumption: χ is even.

Under this assumption, Proposition 5 can be rephrased as stating:

$$\Lambda(\chi, s) = \frac{1}{2}\int_0^\infty \theta(t,\chi) t^{s/2}\,\frac{dt}{t}.$$

We would now like to derive a functional equation for $\theta(t,\chi)$ using the **Poisson summation formula**. Recall that in the proof of the Poisson summation formula we created a periodic function F by summing over all integer translations of f, and then we took the Fourier series of F. After computing the coefficients of the Fourier series we obtained an expression of the form:

$$\sum_{n\in\mathbb{Z}} e^{-\pi(x+n)^2/t} = \sqrt{t}\cdot\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}\cdot e^{2\pi i n x} \tag{2}$$

from which the Poisson summation formula followed by evaluating both sides of (2) at $x = 0$. For our current application, we will follow a different approach: we will evaluate (2) at $x = a/q$ for some a with $0 \leq a \leq q-1$. Then

$$\sum_{n\in\mathbb{Z}} e^{-\pi(a/q+n)^2/t} = \sqrt{t}\cdot\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}\cdot e^{2\pi i n a/q}.$$

Upon replacing t by t/q we get:

$$\sum_{n\in\mathbb{Z}} e^{-\pi(a+qn)^2/tq} = \sqrt{\frac{t}{q}}\cdot\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t/q}\cdot e^{2\pi i n a/q}. \tag{3}$$

We can rewrite the right-hand side of (3) as

$$\sqrt{\frac{t}{q}}\cdot\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t/q}\cdot \psi_a(n) \tag{4}$$

where

$$\begin{aligned}\psi_a : \mathbb{Z}/q\mathbb{Z} &\longrightarrow \mathbb{C}^\times\\ n &\longmapsto e^{2\pi i a n/q}\end{aligned}$$

is an **additive** character modulo q (i.e. $\psi_a(m+n) = \psi_a(m)\cdot\psi_a(n)$).

QUESTION 7. How do we replace the additive character ψ_a in the sum (4) by the multiplicative character χ?

The key idea is to express χ as a linear combination of additive characters ψ_a. This can be accomplished by doing Fourier analysis on the group $(\mathbb{Z}/q\mathbb{Z}, +)$. In fact, the space $L^2(\mathbb{Z}/q\mathbb{Z};\mathbb{C})$ forms a finite dimensional Hilbert space with bilinear form:

$$\langle f, g\rangle = \frac{1}{q}\sum_{x\in\mathbb{Z}/q\mathbb{Z}} f(x)\overline{g(x)}.$$

The additive characters $\{\psi_0, \psi_1, \ldots, \psi_{q-1}\}$ give an orthonormal basis for this Hilbert space so we can write:

$$\chi(n) = \sum c_a(\chi)\psi_a(n), \qquad \forall n \in \mathbb{Z}/q\mathbb{Z}.$$

as a function in $L^2(\mathbb{Z}/q\mathbb{Z};\mathbb{C})$. Then all we need to do is figure out the coefficients $c_a(\chi)$. This will be accomplished through the theory of Gauss sums, after which we will turn our attention to the case of χ an odd character, where the θ function approach does not work.