Math 726: L-functions and modular forms Fall 2011

Lecture 37 : The proof of a p-adic class number formula.

Instructor: Henri Darmon *Notes written by: Francesc Castella*

We will end this course with the proof of a p-adic analogue of the class number formula of Dirichlet's that was proven in the last Lecture. Before we can state it, we need to introduce a p-adic analogue of the classical logarithm.

LEMMA 1. *There exists a unique function*

$$\log_p : \mathbb{G}_m(\mathcal{O}_{\mathbb{C}_p}) \longrightarrow \mathbb{C}_p$$

satisfying the following two properties:

1. $\log_p(1+t) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{t^n}{n}$, *for all* $t \in \mathcal{O}_{\mathbb{C}_p}^\times$ *with* $|t| < 1$;
2. $\log_p(ab) = \log_p(a) + \log_p(b)$, *for all* $a, b \in \mathcal{O}_{\mathbb{C}_p}^\times$.

Proof. Consider the reduction map

$$\mathbb{G}_m(\mathcal{O}_{\mathbb{C}_p}) \longrightarrow \mathbb{G}_m(\overline{\mathbb{F}}_p).$$

Given $x \in \overline{\mathbb{F}}_p$, it lies in a finite extension of $\mathbb{F}_p$, and so there exists a $q = p^m$ such that $x^q = x$. The *Teichmüller lift* of x is then defined by

$$\xi_x := \lim_{j\to\infty} \tilde{x}^{q^j} \in \mathcal{O}_{\mathbb{C}_p}$$

where $\tilde{x}$ denotes an arbitrary lift of x in $\mathcal{O}_{\mathbb{C}_p}$ under the reduction map. Note that

- the expression ξ_x defines an element in $\mathcal{O}_{\mathbb{C}_p}$, since $\{\tilde{x}^{q^j}\}_j$ is a Cauchy sequence and $\mathcal{O}_{\mathbb{C}_p}$ is complete;
- the value $\xi_x \in \mathcal{O}_{\mathbb{C}_p}$ is independent of the chosen lift $\tilde{x}$; and
- it satisfies $\xi_x^q = \xi_x$, as is readily seen from its definition.

Now any $x \in \mathcal{O}_{\mathbb{C}_p}^\times$ can be written as

$$x = \xi_x \cdot (1+t), \qquad \text{for some } t \in \mathcal{O}_{\mathbb{C}_p} \text{ with } |t| < 1,$$

(indeed, x and ξ_x have the same reduction) and we are led to set

$$\log_p(x) = \log_p(\xi_x) + \log_p(1+t) = \log_p(1+t) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{t^n}{n}.$$

This gives a function with the desired properties, and uniqueness follows immediately. □

In order to evaluate the expression

$$L_p(1,\chi) = \int_{\mathbb{Z}_p^\times} x^{-1} d\mu_\chi(x)$$

we would need to consider the "measure" $x^{-1}\mu_\chi$ on $\mathbb{Z}_p$. This has a singularity at $x = 0$, leading to difficulties when we try to integrate the constant function $1 = \binom{x}{0}$ against it. But for $n \geq 1$, the integration of $\binom{x}{n}$ against $x^{-1}\mu_\chi$ on $\mathbb{Z}_p$ causes no problem, and in fact

$$\int_{\mathbb{Z}_p} \binom{x}{n} x^{-1} d\mu_\chi(x) = \frac{1}{n} \int_{\mathbb{Z}_p} \binom{x-1}{n-1} d\mu_\chi(x). \tag{1}$$

These considerations lead us to introduce the *regularised Amice transform*

$$\widetilde{\mathcal{A}}_{x^{-1}\mu_\chi}(T) = \sum_{n=1}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{n} x^{-1} d\mu_\chi(x) \right) T^n = \int_{\mathbb{Z}_p} \left[(1+T)^x - 1\right] x^{-1} d\mu_\chi(x).$$

(Notice that the "regularisation" consists in removing the constant term in what would correspond to the usual Amice transform.) We observe the following:

- As we see from (1), writting $\widetilde{\mathcal{A}}_{x^{-1}\mu_\chi}(T) = \sum_{n=1}^{\infty} \lambda_n T^n$, the coefficients λ_n lie in $\frac{1}{n}\mathcal{O}_{\mathbb{C}_p}$, and hence $\widetilde{\mathcal{A}}_{x^{-1}\mu_\chi}(T) \notin \mathbb{C}_p \otimes \mathcal{O}_{\mathbb{C}_p}[[T]]$ since the λ_n have unbounded denominators. Nevertheless, the power series $\widetilde{\mathcal{A}}_{x^{-1}\mu_\chi}(T)$ still converges for $|T| < 1$.

- $\widetilde{\mathcal{A}}_{x^{-1}\mu_\chi}(T)$ is a "primitive" of $\mathcal{A}_{\mu_\chi}(T)$, in the sense that

$$(1+T)\frac{d}{dT}\widetilde{\mathcal{A}}_{x^{-1}\mu_\chi}(T) = \mathcal{A}_{\mu_\chi}(T). \tag{2}$$

  Indeed:

$$\begin{aligned}
(1+T)\frac{d}{dT}\widetilde{\mathcal{A}}_{x^{-1}\mu_\chi}(T) &= (1+T)\frac{d}{dT}\sum_{n=1}^{\infty}\left(\frac{1}{n}\int_{\mathbb{Z}_p}\binom{x-1}{n-1}d\mu_\chi(x)\right)T^n \\
&= (1+T)\sum_{n=1}^{\infty}\int_{\mathbb{Z}_p}\binom{x-1}{n-1}d\mu_\chi(x)T^{n-1} \\
&= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\left[\binom{x-1}{n} - \binom{x-1}{n-1}\right]d\mu_\chi(x)T^n \\
&= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{x}{n}d\mu_\chi(x)T^n \\
&= \mathcal{A}_{\mu_\chi}(T).
\end{aligned}$$

Theorem 1. *There exists a constant $C \in \mathbb{C}_p$ such that*

$$\widetilde{\mathcal{A}}_{x^{-1}\mu_\chi}(T) = \frac{-1}{\tau(\chi^{-1})} \sum_{a=0}^{D-1} \chi^{-1}(a) \cdot \log_p(\zeta_D^a(1+T)-1) + C. \tag{3}$$

Proof. Letting $H(T)$ denote the RHS of (3) (the constant is irrelevant here), we see that

- $H(T)$ defines a convergent power series for $|T| < 1$, since

$$\begin{aligned}\log_p(\zeta_D^a(1+T)-1) &= \log_p(\zeta_D^a - 1 + \zeta_D^a T) \\ &= \log_p((\zeta_D^a-1)(1+\frac{\zeta_D^a T}{\zeta_D^a - 1})) = \log_p(\zeta_D^a - 1) + \log_p(1+\frac{\zeta_D^a T}{\zeta_D^a-1}) \\ &= \log_p(\zeta_D^a - 1) + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\zeta_D^{an}}{n(\zeta_D^a-1)^n}T^n.\end{aligned}$$

- $H(T)$ satisfies

$$(1+T)\frac{d}{dT}H(T) = \mathcal{A}_{\mu_\chi}(T).$$

  This follows from a straighforward computation that is left as an exercise.

These two facts together with (2), imply that $\widetilde{\mathcal{A}}_{x^{-1}\mu_\chi}(T)$ and $H(T)$ differ by a constant. □

It remains to compute $\widetilde{\mathcal{A}}_{\mathrm{res}_{\mathbb{Z}_p^\times}(x^{-1}\mu_\chi)}(T)$. In fact, since $\mathrm{res}_{\mathbb{Z}_p^\times}(x^{-1}\mu_\chi)$ is a honest measure, the regularisation process is no longer neeeded, and so reflected in the following.

Proposition 1.

$$\mathcal{A}_{\mathrm{res}_{\mathbb{Z}_p^\times}(x^{-1}\mu_\chi)}(T) = \frac{-1}{\tau(\chi^{-1})}\left(1 - \frac{\chi(p)}{p}\right)\sum_{a=0}^{D-1}\chi^{-1}(a)\cdot\log_p(\zeta_D^a(1+T)-1).$$

Proof. First, from Lemma 2 in Lecture 35, giving the formula for the Amice transform of a measure restricted to $p\mathbb{Z}_p$ in terms of that of the original measure, we have

$$\widetilde{\mathcal{A}}_{\mathrm{res}_{p\mathbb{Z}_p}(x^{-1}\mu_\chi)}(T) = \frac{1}{p}\sum_{\xi \in \boldsymbol{\mu}_p}\widetilde{\mathcal{A}}_{x^{-1}d\mu_\chi}(\xi(1+T)-1).$$

Using this, we compute

$$
\begin{aligned}
\mathcal{A}_{\mathrm{res}_{\mathbb{Z}_p^\times}(x^{-1}\mu_\chi)}(T) &= \widetilde{\mathcal{A}}_{x^{-1}\mu_\chi}(T) - \widetilde{\mathcal{A}}_{\mathrm{res}_{p\mathbb{Z}_p}(x^{-1}\mu_\chi)}(T) \\
&= \frac{-1}{\tau(\chi^{-1})}\sum_{a=0}^{D-1}\chi^{-1}(a)\left(\log_p(\zeta_D^a(1+T)-1) - \frac{1}{p}\sum_{\xi\in\boldsymbol{\mu}_p}\log_p(\zeta_D^a\xi(1+T)-1)\right) \\
&= \frac{-1}{\tau(\chi^{-1})}\sum_{a=0}^{D-1}\chi^{-1}(a)\left(\log_p(\zeta_D^a(1+T)-1) - \frac{1}{p}\log_p(\zeta_D^{ap}(1+T)^p-1)\right) \\
&= \frac{-1}{\tau(\chi^{-1})}\left(\sum_{a=0}^{D-1}\chi^{-1}(a)\cdot\log_p(\zeta_D^a(1+T)-1) - \frac{1}{p}\sum_{a=0}^{D-1}\chi^{-1}(a)\cdot\log_p(\zeta_D^{ap}(1+T)^p-1)\right) \\
&= \frac{-1}{\tau(\chi^{-1})}\left(\sum_{a=0}^{D-1}\chi^{-1}(a)\cdot\log_p(\zeta_D^a(1+T)-1) - \frac{\chi(p)}{p}\sum_{a=0}^{D-1}\chi^{-1}(a)\cdot\log_p(\zeta_D^a(1+T)-1)\right) \\
&= \frac{-1}{\tau(\chi^{-1})}\left(1-\frac{\chi(p)}{p}\right)\sum_{a=0}^{D-1}\chi^{-1}(a)\cdot\log_p(\zeta_D^a(1+T)-1),
\end{aligned}
$$

where the penultimate equality follows after replacing ap by a in the right terms in the preceding equality, so that $\chi(a)$ gets replaced by $\chi(a)\chi^{-1}(p)$. The result follows. $\square$

As an immediate consequence, the proof of the p-adic analogue of Dirichlet's class number formula follows. (Cf. Theorem 2 from Lecture 36.)

THEOREM 2. *Let χ be a primitive Dirichlet character of conductor $D > 1$ prime to p. Then*

$$L_p(1,\chi) = \frac{-1}{\tau(\chi^{-1})}\left(1-\frac{\chi(p)}{p}\right)\sum_{a=0}^{D-1}\chi^{-1}(a)\cdot\log_p(1-\zeta_D^a).$$

Proof. Indeed, Proposition 1 gives the last of the following equalities

$$
\begin{aligned}
L_p(1,\chi) = \int_{\mathbb{Z}_p^\times} x^{-1}d\mu_\chi(x) &= \mathcal{A}_{\mathrm{res}_{\mathbb{Z}_p^\times}(x^{-1}\mu_\chi)}(T)\Big|_{T=0} \\
&= \frac{-1}{\tau(\chi^{-1})}\left(1-\frac{\chi(p)}{p}\right)\sum_{a=0}^{D-1}\chi^{-1}(a)\cdot\log_p(1-\zeta_D^a).
\end{aligned}
$$

$\square$

Some concluding remarks are in order. The above class number formula can be seen as a manifestation of a rather elusive connection between two types of objects of a completely different nature, namely *special values* of L-functions –the values of the Dirichlet L-function $L(s,\chi)$, as interpolated by $L_p(s,\chi)$–, and *special elements* of arithmetic content –the cyclotomic units $1-\zeta_D^a$, in the form of their p-adic logarithms.

In the literature one can find p-adic analogues of essentially each of the types of L-functions that have been treated in this course: constructions due to Mazur and Swinnerton-Dyer *et.al.*, corresponding to the Hecke L-functions $L(f, s)$; to Hida *et.al.*, corresponding to Rankin-Selberg L-functions like $L(f \otimes g, s)$; etc.. And for each of these other types of p-adic L-functions, the connections that arise with arithmetic in the form of generalised class number formulae is currently a broad and deep area of great mathematical interest, with many of its gems still awaiting further exploration.