Math 726: L-functions and modular forms — Fall 2011

Lectures 34 : Values of zeta functions

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Recall from last lecture

$D(\mathbb{Z}_p, \mathbb{Q}_p)$ = space of $\mathbb{Q}_p$-valued measures on $\mathbb{Z}_p$.

Definition 1. Given $\mu \in D(\mathbb{Z}_p, \mathbb{Q}_p)$, the **Amice transform** of μ is the power series;

$$\begin{aligned} A_\mu(T) &:= \sum_{n=0}^{\infty} \mu \binom{x}{n} T^n \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \binom{x}{n} d\mu(x) T^n \\ &= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \binom{x}{n} d\mu(x) T^n \\ &= \int_{\mathbb{Z}_p} (1+T)^x d\mu(x) \end{aligned}$$

Remark 1. $\mu \to A_\mu(T)$ gives an isomorphism $D(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathbb{Q}_p \otimes \mathbb{Z}_p[[T]]$

Example 2. **Zeta measure**
Write $\frac{1}{T} - \frac{a}{(1+T)^a - 1}$ for some $a \in \mathbb{Z}_p^*$ (for example we can choose $a = 2$ if p is odd)

Lemma 1. $\frac{1}{T} - \frac{a}{(1+T)^a-1}$ *belongs to* $\mathbb{Z}_p[[T]]$

Proof: $\frac{1}{T} - \frac{a}{(1+T)^a-1} = \frac{1}{T} - \frac{a}{\sum_{j=1}^{\infty} \binom{a}{j} T^j} = \frac{1}{T}(1 - \frac{1}{\sum_{j=1}^{\infty} a^{-1} \binom{a}{j} T^{j-1}} \in \frac{1}{T}(T\mathbb{Z}_p[[T]]) = \mathbb{Z}_p[[T]]$
Using the isomorphism, $D(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathbb{Q}_p \otimes \mathbb{Z}_p[[T]]$, there is a measure μ_a such that:

$$A_{\mu_a}(T) = \frac{1}{T} - \frac{a}{(1+T)^a - 1}$$

This measure is a **Zeta measure**. (We'll see after why it is called a Zeta measure)

Moments of a measure:

The **nth-moment** of a measure μ is : $\mu(x^n) = \int_{\mathbb{Z}_p} x^n d\mu(x)$

THEOREM 3. $\int_{\mathbb{Z}_p} x^n d\mu_a(x) = (-1)^n(1-a^{1+n})\zeta(-n)$,$\forall n \geq 0, n \in \mathbb{Z}$

REMARK 2. If $h(t) = \sum \lambda_n(x)t^n$ is a power series with coefficients in $\mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$, we can define:
$\int_{\mathbb{Z}_p} h(t)d\mu(x) = \sum_{n=0}^{\infty}(\int_{\mathbb{Z}_p} \lambda_n(x)\mu(x))t^n \in \mathbb{Q}_p[[T]]$

EXAMPLE 4. $h(t) = (1+t)^x = 1 + xt + \binom{x}{2}t^2 + ... + \binom{x}{n}T^n + ...$ then:
$\int_{\mathbb{Z}_p} h(t)d\mu(x) := \sum_{n=0}^{\infty} \mu\binom{x}{n}t^n = A_\mu(t) \in \mathbb{Q}_p \otimes \mathbb{Z}_p[[T]]$

EXAMPLE 5. $h(t) = e^{tx} = 1 + xt + ... + \frac{x^n t^n}{n!} + ...$ then:
$\int_{\mathbb{Z}_p} h(t)d\mu(x) := \sum_{n=0}^{\infty} \mu(x^n)\frac{t^n}{n!} \in \mathbb{Q}_p[[T]]$

Proof of the theorem:

$$
\begin{aligned}
\int_{\mathbb{Z}_p} x^n d\mu_a(x) &= \left[\left(\frac{d}{dt}\right)^{(n)}\left(\int_{\mathbb{Z}_p} e^{tx}d\mu_a(x)\right)\right]_{|t=0} \\
&= \left[\left(\frac{d}{dt}\right)^{(n)}\left(\int_{\mathbb{Z}_p} (1+T)^x d\mu_a(x)\right)\right]_{|t=0} \quad \text{where } T = e^t - 1 \\
&= \left[\left(\frac{d}{dt}\right)^{(n)}(A_{\mu_a}(T))\right]_{|t=0} \\
&= \left[\left(\frac{d}{dt}\right)^{(n)}(A_{\mu_a}(e^t-1))\right]_{|t=0} \\
&= \left[\left(\frac{d}{dt}\right)^{(n)}\left(\frac{1}{e^t-1} - \frac{a}{e^{at}-1}\right)\right]_{|t=0} \\
&= f_a^{(n)}(0) \text{ where } f_a = \frac{1}{e^t-1} - \frac{a}{e^{at}-1} \\
&= (-1)^n L(f_a, -n) \text{ where } L(f_a, s) := \frac{1}{\Gamma(s)}\int_0^{\infty} f_a(t)t^s\frac{dt}{t}
\end{aligned}
$$

But, $L(f_a, s) = \frac{1}{\Gamma(s)}\int_0^{\infty}\left[(e^{-t} + e^{-2t} + ...) - a\left(e^{-at} + e^{-2at} + ...\right)\right]t^s\frac{dt}{t}$
$= \zeta(s) - a^{1-s}\zeta(s)$ by making the changes of variable $u = nt$ and $v = ant$.
$= (1-a^{1-s})\zeta(s)$

Finally,

$$\boxed{\int_{\mathbb{Z}_p} x^n d\mu_a(x) = (-1)^n(1-a^{1-n})\zeta(-n)}$$

$\square$

Corollary 1. *If $n_1 \equiv n_2$ (mod $(p-1)p^k$) and $n_1, n_2 \geq k+1$, then:*

$$(1-a^{1+n_1})\zeta(-n_1) \equiv (1-a^{1+n_2})\zeta(-n_2) \ (mod\ p^{k+1})$$

<u>Proof:</u>

If $n_1 \equiv n_2$ (mod $(p-1)p^k$) and $n_1, n_2 \geq k+1$, then $\mathrm{v}_p(x^{n_1} - x^{n_2}) \geq k+1$ because:

$x^{n_1} \equiv x^{n_2}$ (mod p^{k+1}) $\forall x \in \mathbb{Z}_p^*$ using $n_1 \equiv n_2$ (mod $(p-1)p^k$) and the fact that $\left(\mathbb{Z}/p^{k+1}\mathbb{Z}\right)^*$ has order $(p-1)p^k$

$x^{n_1} \equiv x^{n_2} \equiv 0$ (mod p^{k+1}) $\forall x \in p\mathbb{Z}_p$ using $n_1, n_2 \geq k+1$.

Hence, $\mathrm{v}_p(\mu_a(x^{n_1}) - \mu_a(x^{n_2})) \geq k+1$ since μ_a sends $\mathcal{C}(\mathbb{Z}_p, \mathbb{Z}_p)$ to $\mathbb{Z}_p$. (since $\frac{1}{T} - \frac{a}{(1+T)^a-1} \in \mathbb{Z}_p[[T]]$) $\square$.

But following the proof, $n \to (1-a^{1-n})\zeta(-n)$ does NOT extend to a continuous function on $\mathbb{Z}_p \times \mathbb{Z}/p\mathbb{Z}$ (the problem comes from the extra hypothesis on n_1 and n_2).

Question 6. How can we transform $\int_{\mathbb{Z}_p} x^n d\mu_a(x)$ in such a way that we could extend ζ to $\mathbb{Z}_p$?

A related problem is the fact that we cannot define the nth-power of x for all $n \in \mathbb{Z}_p$ and all $x \in \mathbb{Z}_p$.

For example, the definition $x^n := \sum_{k=0}^{\infty} \binom{n}{k}(x-1)^k$ only works when $x \in 1 + p\mathbb{Z}_p$.

Furthermore, one can give a good definition for $x \to x^n$, $\forall x \in \mathbb{Z}_p^*$ and $\forall n \in \mathbb{Z}_p$

In that sense, $x \to \mathbf{1}_{\mathbb{Z}_p^*}.x^n$ is a continuous function on $\mathbb{Z}_p$. ($\mathbf{1}_{\mathbb{Z}_p^*}$ is the characteristic function on $\mathbb{Z}_p^*$ and is CONTINUOUS)

So we can take its μ_a measure:

$$\int_{\mathbb{Z}_p} \mathbf{1}_{\mathbb{Z}_p^*}.x^n d\mu_a(x) := \int_{\mathbb{Z}_p^*} x^n d\mu_a(x)$$

GOAL: Relate $\int_{\mathbb{Z}_p^*} x^n d\mu_a(x)$ to $\zeta(-n)$.(simple factor)