Math 726: L-functions and modular forms Fall 2011

Lecture 30 : Serre-Deligne theorem and special values of L-functions.

Instructor: Henri Darmon *Notes written by: Francesca Gala*

Comments on the proof of Serre-Deligne Theorem.

In the last lecture we showed the existence, for every $\lambda \in \Sigma = \{\lambda \triangleleft \mathcal{O}_{K_f} | \mathcal{O}_{K_f}/\lambda \cong \mathbb{F}_\ell\}$, of a representation ρ_f associated to the eigenform $f \in S_1(\Gamma_0(D), \epsilon)$:

$$\rho_f : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(K_{f,\lambda}) \hookrightarrow \mathrm{GL}_2(\mathbb{C}),$$

which satisfies $\mathrm{char}(\rho_f(\mathrm{Frob}_p)) = x^2 - a_p x + \epsilon(p)$.

Ramification. By construction ρ is unramified outside of $D \cdot \ell$, but in fact if we choose another $\lambda' \in \Sigma$ above a prime $\ell' \neq \ell$, then we obtain another representation ρ'_f:

$$\rho'_f : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(K_{f,\lambda'}) \hookrightarrow \mathrm{GL}_2(\mathbb{C}),$$

which is now unramified outside $D \cdot \ell'$ and is such that $\rho = \rho'$, by semisimplicity and Chebotarev density theorem. Hence ρ is unramified outside of the primes dividing D.

Irreducibility. We have the following proposition.

PROPOSITION 1. *Let $f \in S_1(D, \epsilon)$ and $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ the associated representation. Then ρ_f is irreducible.*

Proof. Consider $L(f \otimes \overline{f}, s)$. Since f is a cusp form Rankin's method shows that this L-function has a simple pole at $s = 1$ and $L(f \otimes \overline{f}, s) = L(\rho_f \otimes \overline{\rho_f}, s)$.
Suppose that $\rho_f = \chi_1 \oplus \chi_2$. Since ρ_f is odd we can assume without loss of generality that $\chi_1(-1) = 1$ and $\chi_2(-1) = -1$. We have the equality:

$$L(f \otimes \overline{f}, s) = \zeta(s)^2 L(\chi_1\overline{\chi_2}, s) L(\chi_2\overline{\chi_1}, s)$$

up to finitely many Euler factors, which are non-zero at $s = 1$.
$\zeta(s)^2$ has a double pole at $s = 1$ and $L(\chi_1\overline{\chi_2}, 1)L(\chi_2\overline{\chi_1}, 1) \neq 0$, so we have a contradiction since we had proved that $\mathrm{ord}_{s=1} L(f \otimes \overline{f}, s) = -1$ □

Summary of the proof of Serre-Deligne.
Step 1. Construction of $f_\ell = f E_{\ell-1} \in S_\ell(D, \epsilon)$ **modular form of weight** ℓ, not necessarily

an eigenform, but for which the reduction modulo ℓ is an eigenform, since $E_{\ell-1} \equiv 1 \bmod \ell$.
Step 2. By means of the **Serre Deligne lifting theorem**, we obtain an eigenform $\tilde{f}_\ell \in S_\ell(D, \epsilon)$.
Step 3. We use results of **Eichler-Shimura theory** to associate to $\tilde{f}_\ell$ a compatible system of λ-adic representations

$$\rho_{f,\lambda} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\tilde{K_\lambda}),$$

where in this case $\mathrm{char}(\rho_{f,\lambda}(\mathrm{Frob}_p)) = x^2 - a_p(\tilde{f}_\ell)x + p^{\ell-1}\epsilon(p)$.
Step 4. We now reduce the $\rho_{f,\lambda}$ to a system of semisimple representations modulo λ:

$$\overline{\rho_{f,\lambda}} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathcal{O}_{\tilde{K_\lambda}}/\lambda).$$

Step 5. We prove, using the **Rankin-Selberg method**, that $\overline{\rho_{f,\lambda}}(G_{\mathbb{Q}})$ are bounded independently of ℓ. Then we conclude that for λ large enough $\overline{\rho_{f,\lambda}}$ can be lifted to a complex representation.

Remark 1. As a by-product of the proof of the Serre-Deligne theorem we obtained a 'very nice' analytic estimate on the coefficients of the eigenform f, that we did not have a priori.

We are now able to understand the following types of L-functions.

1. *The Riemann zeta function* $\zeta(s)$ and the *Dirichlet L-function* $L(\chi, s)$ attached to a complex character $\chi : G_{\mathbb{Q}} \to \mathbb{C}^\times$.
2. The L-function $L(\rho, s)$ attached to certain (very special) two-dimensional Artin representations of $G_{\mathbb{Q}}$: those arising from cusp forms of weight 1.
3. The L-functions $L(V, s)$ attached to certain (very special) 2-dimensional compatible systems of ℓ-adic representations, namely those arising from modular forms of weight $k \geq 2$.
The PROTOTYPICAL EXAMPLE of this case is the representation attached to a newform form $f \in S_2(\Gamma_0(N))$ with rational Fourier coefficients. We have shown that in this case ρ_f arises from $\varprojlim_n J_0(N)[\ell^n]$ and that we can associate to f an elliptic curve E such that $L(f, s) = L(E, s)$.

This examples are not so special!
In fact we have the following results:
- (Wiles et al, 1995) Every elliptic curve $E/\mathbb{Q}$ arises from a newform $f \in S_2(\Gamma_0(N))$ with rational Fourier coefficients.
- (Taylor, Khare-Winterberg, Kisin, 1995-2006) If $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ is an odd, irreducible representation of $G_{\mathbb{Q}}$ then there exists $f \in S_1(\Gamma_0(D), \epsilon)$ which satisfies $L(\rho, s) = L(f, s)$.

Special values of L-functions.

We would now like to investigate on the special values of $L(V,s)$ at $s=j$.

EXAMPLE 1 (CLASS NUMBER FORMULA). Consider the Dedekind $\zeta_K(s)$ function. Recall from the first lecture the following formulas:

$$\operatorname{res}_{s=1}(\zeta_K(s)) = \frac{2^{r_1}(2\pi)^{r_2}h_K R_k}{\frac{1}{2}|(\mathcal{O}_K^\times)_{\mathrm{tors}}|\sqrt{\operatorname{Disc}(K)}},$$

where R_K is the *regulator* of the field K and moreover

$$\zeta_K(s) \sim \frac{2^{r_1}(2\pi)^{r_2}h_K R_k}{\frac{1}{2}|(\mathcal{O}_K^\times)_{\mathrm{tors}}|}.$$

DEFINITION 2. Two compatible systems V_1 and V_2 are said to be congruent modulo n if the characteristic polynomials of Frobenius which lie in $\mathbb{Z}[x]$ are congruent modulo n.

QUESTION 3. If $V_1 \equiv V_2 \bmod n$, when can we conclude that $L(V_1,0) = L(V_2,0) \bmod n$?

EXAMPLE 4. Let us consider the representations $\rho_1 := V_1 = \mathbb{Z}(j_1)$ and $\rho_2 := V_2 = \mathbb{Z}(j_2)$, recall that these representations arise from compatible systems of representations which satisfy $\rho_1(\mathrm{Frob}_\ell) = \ell^{j_1}$ and $\rho_2(\mathrm{Frob}_\ell) = \ell^{j_2}$. Then

$$L(\rho_1, s) = \zeta(s+j_1)$$

and

$$L(\rho_2, s) = \zeta(s+j_2).$$

Fix p^m, then $V_1 \equiv V_2 \bmod p^m$ if and only if $j_1 \equiv j_2 \bmod (p-1)p^{m-1}$. Is it then true that $\zeta(j_1) \equiv \zeta(j_2)$?

During the next lecture we are going to exploit another integral representation of the zeta function, different from the one we used to determine analytic continuation, to find information on the values of zeta at negative integers. We will then prove the following:

THEOREM 5. *If $n \in \mathbb{Z}_{\geq 0}$, then $\zeta(-n) \in \mathbb{Q}$ and is non zero if and only if n is odd.*

The proof of this theorem will follow by relating zeta to the Mellin transform of the function of rapid decay

$$f(t) = \frac{t}{e^t - 1} = t\left(\frac{e^{-t}}{1-e^{-t}}\right) = t(e^{-t} + e^{-2t} + e^{-3t} + \cdots). \tag{1}$$

Let us be more precise.

Definition 6. For any function $f : \mathbb{R}^{\geq 0} \to \mathbb{C}$, which is smooth and of rapid decay, we associate an L-function

$$L(f, s) := \frac{1}{\Gamma(s)} \int_0^\infty f(t) t^s \frac{dt}{t}.$$

Lemma 1. *If* $f(t) = \frac{t}{e^t - 1}$ *then* $\zeta(s) = \frac{1}{s-1} L(f, s-1)$.

Proof. The result follows from a direct computation using equality (1). □