Math 726: L-functions and modular forms — Fall 2011

Lecture 3: Fourier transforms and Poisson summation

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In the last lecture we showed how to derive the functional equation of the Riemann ζ function, by letting

$$\Lambda(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

and then by showing

$$\Lambda(s) = \Lambda(1-s).$$

There are two key steps in this proof:

(I) Express $\Lambda(s)$ as a Mellin transform $\Lambda(s) = M(\omega)(s)$, where

$$\omega(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 t}$$

and $t \in \mathbb{R}_{>0}$, to obtain an integral representation of $\Lambda(s)$.

(II) Exploit the identity

$$\theta\left(\frac{1}{x}\right) = \sqrt{x}\cdot\theta(x)\ , \quad x \in \mathbb{R}_{>0} \tag{1}$$

where

$$\theta(x) := \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} = 2\omega(x) + 1$$

The first step was proved in the previous lecture, whereas this lecture will be concerned with the proof of the identity (1).

We first need to recall some notions from Fourier analysis. Let $f : \mathbb{R} \to \mathbb{C}$ be an integrable (i.e. L^2) function.

Definition 1. The **Fourier transform** of f is the function $\widehat{f} : \mathbb{R} \to \mathbb{C}$ given by

$$\widehat{f}(s) = \int_{\mathbb{R}} e^{-2\pi i s t} f(t)\, dt.$$

Remark 2. The notion of a Fourier transform makes sense for any locally compact topological group G. If $\widehat{G}$ is the space of characters $\chi : G \to S^1$, then the Fourier transform can be seen as a map $L^2(G) \to L^2(\widehat{G})$ by sending $f \mapsto \widehat{f}(\chi) = \int_G \overline{\chi}(t) f(t)\, dt$. When $G = \mathbb{R}$, all the characters of G are of the form $\chi_s(t) = e^{2\pi i s t}$ for $s \in G$.

Although we can define Fourier transforms for any function $f \in L^2(\mathbb{R})$, we would like to restrict our attention to a special class of integrable functions on which the process of taking Fourier transforms can be iterated.

DEFINITION 3 (Schwartz function). f is a **Schwartz function** if f is smooth (i.e. C^∞) and if it is of rapid decay (i.e. $|f(x)| \ll |x|^{-N}$ as $x \to \infty$ for all N).

A Schwartz function is clearly integrable, so we can take its Fourier transform.

LEMMA 4. *The Fourier transform preserves the space of Schwartz functions. Moreover:*

(a) $\hat{\hat{f}} = f(-t)$.

(b) $\widehat{f * g}\,(s) = \widehat{f}(s) * \widehat{g}(s)$ *where* $f * g\,(t) = \int_{-\infty}^{\infty} f(x)g(t-x)\,dx$ *is the* **convolution** *of* f *and* g.

(c) $\widehat{f(\lambda t)}(s) = \frac{1}{\lambda}\widehat{f}\left(\frac{s}{\lambda}\right)$ *for any* $\lambda \in \mathbb{R}$.

In particular, we will use property (c), which we prove below.

Proof of (c).

$$\begin{aligned}
\widehat{f(\lambda t)}(s) &= \int_{\mathbb{R}} e^{-2\pi i s t} f(t\lambda)\,dt \\
&= \int_{\mathbb{R}} e^{-2\pi i s u/\lambda} f(u)\frac{du}{\lambda} \qquad (\text{use } u = t\lambda) \\
&= \frac{1}{\lambda} \cdot \widehat{f}\left(\frac{s}{\lambda}\right)
\end{aligned}$$

□

We will also make use of the following important theorem.

THEOREM 5 (**Poisson summation formula**). *Let* $f : \mathbb{R} \to \mathbb{C}$ *be a Schwartz function. Then*

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \widehat{f}(n).$$

Proof. Consider the function $F(x) = \sum_{n \in \mathbb{Z}} f(x+n)$. This is a periodic function of period 1, therefore we can take its Fourier series expansion:

$$F(x) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x}$$

where

$$a_n = \int_0^1 F(x)e^{-2\pi inx}\,dx = \int_0^1 \sum_{m\in\mathbb{Z}} f(x+m)e^{-2\pi inx}$$
$$= \sum_{m\in\mathbb{Z}} \int_0^1 f(x+m)e^{-2\pi inx}\,dx$$
$$= \sum_{m\in\mathbb{Z}} \int_0^1 f(x+m)e^{-2\pi in(x+m)}\,d(x+m)$$
$$= \int_{-\infty}^{\infty} f(t)e^{-2\pi int}\,dt = \widehat{f}(n).$$

Therefore:

$$\sum_{n\in\mathbb{Z}} f(x+n) = F(x) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{2\pi inx}$$

and the result follows by evaluating at $x = 0$. □

Going back to the proof of identity (1), consider now the Gaussian $e^{-\pi t^2}$.

Proposition 6. *The Schwartz function $g(t) = e^{-\pi t^2}$ is its own Fourier transform.*

Proof.

$$\widehat{g}(s) = \int_{\mathbb{R}} e^{-2\pi ixs}\cdot e^{-\pi x^2}\,dx$$
$$= \int_{\mathbb{R}} e^{-\pi(x^2+2ixs)}\,dx$$
$$= \int_{\mathbb{R}} e^{-\pi((x+is)^2+s^2)}\,dx \qquad \text{(complete the square)}$$
$$= e^{-\pi s^2}\cdot \int_{\mathbb{R}} e^{-\pi(x+is)^2}\,dx$$
$$= e^{-\pi s^2}\cdot \int_{z=is+\mathbb{R}} e^{-\pi z^2}\,dz$$

We claim that the integral $\int_{is+\mathbb{R}} e^{-\pi z^2}\,dz$, which is over the line $is+\mathbb{R}$ parallel to the real line in the complex plane, is the same as $\int_{\mathbb{R}} e^{-\pi x^2}\,dx$. This follows by integrating $e^{-\pi z^2}$ along the sides of rectangles of base $2M$ on the real axis and height s: the integral along the whole perimeter is zero by Cauchy's Theorem, but the integral on the vertical sides tends to zero as $M \to \infty$ (see Figure 1).

Therefore

$$\int_{is+\mathbb{R}} e^{-\pi z^2}\,dz = \int_{\mathbb{R}} e^{-\pi x^2}\,dx.$$

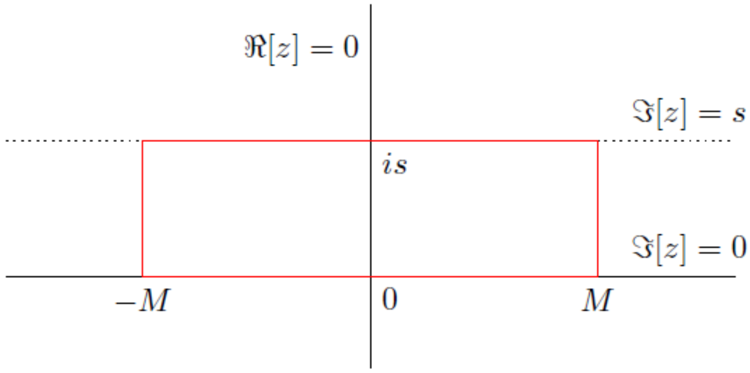

Figure 1: *The integral of $e^{-\pi z^2}$ along the vertical lines tends to 0 as $M \to \infty$.*

To conclude the proof, we need to show that $\int_{\mathbb{R}} e^{-\pi x^2}\, dx = 1$. But this follows from:

$$\begin{aligned}
\int_{\mathbb{R}} e^{-\pi x^2}\, dx &= 2\int_0^\infty e^{-\pi x^2}\, dx \\
&= 2\sqrt{\int_0^\infty e^{-\pi x^2}\, dx \cdot \int_0^\infty e^{-\pi y^2}\, dy} \\
&= 2\sqrt{\int_{r=0}^\infty \int_{\theta=0}^{\pi/2} e^{-\pi r^2} r\, d\theta\, dr} \\
&= 2\sqrt{\frac{\pi}{2}\left[-\frac{1}{2\pi} e^{-\pi r^2}\right]_0^\infty} \\
&= 2\sqrt{\frac{\pi}{2}\frac{1}{2\pi}} = 1
\end{aligned}$$

□

Corollary 7. *Let $f_t(x) = e^{-\pi x^2 t}$. Then*

$$\widehat{f_t}(s) = \frac{1}{\sqrt{t}} e^{-\pi s^2/t}.$$

Proof. Note that $f_t(x) = f_1(\sqrt{t}x)$. By part (c) of Lemma 4 we must have:

$$\widehat{f_t}(s) = \widehat{f_1(\sqrt{t}x)}(s) = \frac{1}{\sqrt{t}}\widehat{f_1}\left(\frac{s}{\sqrt{t}}\right).$$

But $f_1(x)$ is the Gaussian of Proposition 6, therefore $\widehat{f_1}(s) = f_1(s) = e^{-\pi s^2}$. □

We are now ready to prove the identity (1):

$$\theta\left(\frac{1}{x}\right) = \sqrt{x} \cdot \theta(x).$$

Proof of identity (1). Let $f_t(x) = e^{-\pi x^2 t}$ and apply Poisson summation:

$$\begin{aligned}\theta(t) = \sum_{n\in\mathbb{Z}} f_t(n) &= \sum_{n\in\mathbb{Z}} \widehat{f}_t(n) \\ &= \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}} e^{-\pi n^2/t} \\ &= \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}} \theta\left(\frac{1}{t}\right)\end{aligned}$$

□

In particular, the proof of this identity concludes the proof of the functional equation of the ζ function.

REMARK 8. We have defined $\theta(t)$ as a function of $t \in \mathbb{R}_{>0}$ but there is nothing to prevent us from extending its domain to the complex right half-plane $\{z \in \mathbb{C} : \Re[z] > 0\}$. As it is customary in this game, we actually rotate the domain by 90 degrees and define:

$$\widetilde{\theta}(z) := \theta(-iz) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}$$

where z is now a variable in the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \Im[z] > 0\}$. The function $\widetilde{\theta}(z)$ then satisfies the two remarkable transformation properties:

- $\widetilde{\theta}(z+2) = \widetilde{\theta}(z)$ (by definition)
- $\widetilde{\theta}\left(-\frac{1}{z}\right) = \sqrt{-iz} \cdot \widetilde{\theta}(z)$ (by (1))

corresponding, respectively, to the Möbius tranformations $z \mapsto z+2$ and $z \mapsto -1/z$ acting on $\mathbb{H}$. These transformation propeties are typical of **modular forms**. In particular, we say that $\widetilde{\theta}(z)$ is a *modular form of weight 1/2* on the group

$$\Gamma(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z}) : b, c \equiv 0 \mod 2 \text{ and } a, d \equiv 1 \mod 2 \right\}$$

of determinant 1 integer matrices which reduce to the identity modulo 2. We will see in the near future that the function:

$$\Lambda(s) = \int_0^\infty \widetilde{\omega}(it) t^s \,\frac{dt}{t}$$

has a functional equation analogous to the Riemann ζ function, and a similar construction will apply to general modular forms as well.

Going back to our general philosophy that L-functions corresponds to Galois representations, recall that the Riemann ζ function is the L-function associated to the **trivial** Galois representation:

$$\rho_{\text{triv}} : G_{\mathbb{Q}} \longrightarrow \mathbb{C}^{\times}$$

and it can therefore be regarded as the 'simplest' L-function. Next, we will analyze those L-functions that come from 1-dimensional representations

$$\chi : G_{\mathbb{Q}} \longrightarrow \mathbb{C}^{\times}.$$

By the class field theory of $\mathbb{Q}$ (i.e. Kronecker-Weber Theorem) these representations correspond to Dirichlet characters. The attached L-functions are called **Dirichlet L-functions**. Just as we did with the Riemann ζ function, we will try and understand the poles, zeroes and critical values of these new types of L-functions.