Math 726: L-functions and modular forms — Fall 2011

# Lecture 28 : Artin Representations Attached to Forms of Weight One

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We are trying to associate to $f$ a homomorphism $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$. The steps we have taken are:

1. Given $\lambda \lhd \mathcal{O}_{K_p}$ we get $F_\lambda = fE_{\ell-1} \in S_\ell(D, \epsilon)$, an eigenform mod $\lambda$.
2. Prove the Delign-Serre lifting lemma: that there exists $\tilde{F}_{\lambda'} \in S_\ell(D, \epsilon)$ which is an eigenform with coefficients in $K'_\lambda/K_f$ with $a_p(\tilde{F}_{\lambda'}) \cong a_p(f) \pmod{\lambda'}$ for all $p \nmid D$.
3. Now consider the mod $\lambda'$ reduction of the $\lambda'$-adic representation attached to $\tilde{F}_{\lambda'}$:

$$\rho_{f,\lambda'} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}_{K'}/\lambda').$$

The characteristic polynomial of Frobenius mod $\lambda'$ is:

$$\mathrm{char}(\rho_{f,\lambda'}(\mathrm{Frob}_p)) = x^2 - a_p(f)x + \epsilon(p)$$

Let $\bar{\rho}_{f,\lambda}$ denote the semisimplification of $\rho_\lambda$. We will apply the following lemma to $\bar{\rho}_{f,\lambda}$.

Lemma 1. *If $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(L)$ is a semisimple odd representation and the characteristic polynomial of every element is in $K[x]$, then $\rho$ has a conjugate whose image is in $\mathrm{GL}_2(K)$.*

At this stage we have

$$\overline{\rho_{f,\lambda}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}_{K_f}/\lambda)$$

with the desired property.

Let $\Sigma = \{\lambda \lhd \mathcal{O}_{K_f} \,:\, \lambda | \ell \text{ and } \ell \text{ splits completely in } K_f/\mathbb{Q}\}$. By Chebotarev's density theorem $\Sigma$ is infinite. For each $\lambda \in \Sigma$ let $\ell$ be the rational prime lying below it. Furthermore let

$$G_\lambda := \overline{\rho_{f,\lambda}}(G_{\mathbb{Q}}) \subseteq \mathrm{GL}_2(\mathbb{F}_\ell).$$

We wish to bound $|G_\lambda|$ independently of $\ell$.

Definition 1. Fix an integer $X > 0$. A subgroup $G \subset \mathrm{GL}_2(\mathbb{F}_\ell)$ is <u>$X$-sparse</u> if there is a subset $H \subset G$ such that

1. $|H| \geq \frac{3}{4}|G|$, and

2. the elements of $H$ have at most $X$ distinct characteristic polynomials.

LEMMA 2. *There exists an $X > 0$ such that all the groups $G_\lambda$ ($\lambda \in \Sigma$) are $X$-sparse.*

*Proof.* Recall from last week that for all $\eta > 0$ there exists finite set $X_\eta \subset \mathbb{C}$ such that $a_p \in X_\eta$ for all $p$ outside a set $Y_\eta$ of density $\eta$. Take $\eta < \frac{1}{4}$ and set $X = |X_\eta| \operatorname{ord}(\varepsilon)$. We claim that $G_\lambda$ is $X$-sparse. Let

$$H = \bigcup_{p \in Y_\eta} \rho_f(\mathrm{Frob}_p).$$

1. The density of $Y_\eta > \frac{3}{4}$ implies $|H| \geq \frac{3}{4}|G|$ (by Chebotarev density theorem).

2. The number of distinct characteristic polynomials of elements of $H$ is less that $|X_\eta| \operatorname{ord}(\varepsilon)$.

$\square$

DEFINITION 2. A subgroup $G$ of $\mathrm{GL}_2(\mathbb{F}_\ell)$ is semisimple if the underlying 2 dimensional representation of $G$ is semisimple, (that is, either irreducible or a direct sum of 1 dimensional representations).

EXAMPLE 3. The groups $\mathrm{GL}_2(\mathbb{F}_\ell)$ and $\mathrm{SL}_2(\mathbb{F}_\ell)$ are irreducible, hence semisimple.

EXAMPLE 4. If $G = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} | a, b \in \mathbb{F}_\ell^\times \right\}$ (the "split Cartan subgroup"), then $G$ is semisimple and reducible.

EXAMPLE 5. Let $\mathbb{F}_{\ell^2}^\times$ act by left multiplication on $\mathbb{F}_{\ell^2}$ viewed as a $\mathbb{F}_l$ vector space with any choice of basis. Let $G$ be the image of $\mathbb{F}_{\ell^2}^\times$ in $\mathrm{Aut}_{\mathbb{F}_\ell}(\mathbb{F}_{\ell^2}) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$ (a "non-split Cartan subgroup"). Then G is semisimple.

EXAMPLE 6. If $G$ is the normalizer of a Cartan subgroup, then it is semisimple. (Note that the Cartan subgroup then has index 2 in $G$).

EXAMPLE 7. The group $G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\}$ is reducible but not totally reducible, hence $G$ is not semisimple.

REMARK 1. Note that all the $G_\lambda$ are semisimple by construction.

THEOREM 8. *Fix $X$, there exists a constant $A$ (depending on $X$ but not $\ell$) such that $|G| < A$ for all semisimple $X$-sparse subgroups of* $\mathrm{GL}_2(\mathbb{F}_\ell)$.

The proof relies on the following group theory fact.

**Group Theory Fact:** If $G \subset \mathrm{GL}_2(\mathbb{F}_\ell)$ is semisimple, then one of the following cases is true:

1. $G \supseteq \mathrm{SL}_2(\mathbb{F}_\ell)$,
2. $G \subseteq T$ for some Cartan subgroup $T$,
3. $G \subseteq N(T)$ the normalizer of a Cartan subgroup but $G \nsubseteq T$, or
4. the image of $G$ in $PGL_2(\mathbb{F}_\ell)$ is isomorphic to one of $A_4, S_4, A_5$.