Math 726: L-functions and modular forms Fall 2011

# Lecture 27 : Artin Representations Attached to Forms of Weight One

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Recall what we had this morning:

Let $f \in S_1(\Gamma_0(D), \varepsilon)$ with Fourier coefficients in $\mathcal{O}_{K_f} \subseteq K_f$. Given $\lambda \triangleleft \mathcal{O}_{K_f}$ with $\lambda | \ell$ and $\ell > 3$ we considered:

$$F_\lambda = f E_{\ell-1} \in S_\ell(\Gamma_0(N), \varepsilon).$$

Here $F_\lambda$ is an eigenform modulo $\lambda^n$, ie, $T_n(F_\lambda) - a_n(F_\lambda) \in \lambda \mathcal{O}_{K_{f,\lambda}}[[q]]$.

Theorem 1 (Delign-Serre Lifting Lemma). *Given $F_\lambda$ there exists*

1. *a finite extension $K'$ of $K = K_f$,*
2. *an ideal $\lambda' \in \mathcal{O}_{K'}$ such that $\lambda' \cap \mathcal{O}_K = \lambda$, and*
3. *a normalized eigenform $\tilde{F}_{\lambda'} \in S_\ell(\Gamma_0(D), \varepsilon)$ with Fourier coefficients in $\mathcal{O}_K$ such that*

$$a_n(\tilde{F}_{\lambda'}) \cong a_n(F_\lambda) \pmod{\lambda'} \qquad \textit{for all } (n, D) = 1.$$

*Proof.* Let $\mathbb{T}$ be the ring generated by the Hecke operators $T_n$ with $(n, D) = 1$ inside $\mathrm{End}(S_l(\Gamma_0(D), \varepsilon))$.

Structure of $\mathbb{T}$

Basic facts we will need. (For this we will assume $\varepsilon$ is quadratic.)

1. $\mathbb{T}$ is a maximal commutative subring of $M_d(\mathbb{Z})$,. This implies $\mathbb{T}$ is isomorphic to $\mathbb{Z}^d$ as a $\mathbb{Z}$-module (not as a ring).

   If $L$ is any ring then $\mathbb{T}_L = \mathbb{T} \otimes_{\mathbb{Z}} L \cong L^d$ as a $\mathbb{Z}$-module.

2. $\mathbb{T}_{\mathcal{O}_{K,\lambda}}$ is equipped with a homomorphism to $\mathcal{O}_{K,\lambda}$ arising from the mod $\lambda$ eigenform $F_\lambda$.

$$\begin{aligned} \varphi_{F_\lambda} : \mathbb{T}_{\mathcal{O}_{K,\lambda}} &\to (\mathcal{O}_{K,\lambda}/\lambda) = F_{\ell^s} \\ T_n &\mapsto a_n \pmod{\lambda}. \end{aligned}$$

Goal: We want to lift $\varphi_{F_\lambda}$ to a homomorphism $\varphi_{f,\lambda}$ in characteristic 0.

$$\begin{tikzcd} & \mathcal{O}_{K',\lambda'} \arrow[d] \\ \mathbb{T}_{\mathcal{O}_{K,\lambda}} \arrow[r] \arrow[ur, dotted] & \mathcal{O}_{K,\lambda} \end{tikzcd}$$

(We can do this for a larger field $K'$ and $\lambda'$.)

Let $M = \ker(\varphi_{F_\lambda})$. $M$ is a maximal ideal of $\mathbb{T}_{\mathcal{O}_{K,\lambda}}$.

Lemma 1. *The ideal $M$ properly contains a prime ideal $I$ of $\mathbb{T}_{\mathcal{O}_{K,\lambda}}$.*

*Proof.* Consider the injective map

$$\mathbb{T} \otimes_{\mathbb{Z}} \mathcal{O}_L \overset{(\varphi_i)_i}{\hookrightarrow} \mathcal{O}_L^d$$

induced by the eigenvalues of the mod $\lambda$ eigenform $F_\lambda$ Let $I_j := \ker(\varphi_j)$ be the various kernels.

We claim that there exists an $I_j \subset M$. Indeed, suppose $I_j \nsubseteq M$ for all $j$. Then for each $j$ there exists $x_j \in I_j \setminus M$. We then have that the product $x_1 \dots x_d \in I_1 \cap \dots \cap I_d$. But this intersection is trivial as the above map is injective. Thus, $x_1 \cdots x_d = 0$ which contradicts $x_1, \dots, x_d \notin M$. □

Take $\tilde{F}_{\lambda'} \in S_\ell(D, \varepsilon)[I]$ normalized. Then $T_n - a_n(\tilde{F}_{\lambda'}) \in I$, hence $\tilde{F}_{\lambda'}$ is an eigenform and

$$a_n(\tilde{F}_{\lambda'}) \equiv a_n(F_\lambda) \pmod{\lambda'}.$$ □

Theorem 2. *If $f$ is a normalized eigenform in $S_1(\Gamma_0(D), \varepsilon)$, then for each $\lambda \lhd \mathcal{O}_{K_f}$ there exists*

$$\overline{\rho_{f,\lambda}} : G_{\mathbb{Q}} \to \mathrm{GL}_2\left(\mathcal{O}_{K_f}/\lambda\right)$$

*satisfying*

$$\mathrm{char}(\overline{\rho_{f,\lambda}}(\mathrm{Frob}_p)) = x^2 - a_p x + \varepsilon(p).$$

*Proof.* Let $F_\lambda = E_{\ell-1} f$ and let $\tilde{F}_{\lambda'}$ be the eigenform obtained from $F_\lambda$ be the Deligne-Serre lifting lemma. Recall that $\tilde{F}_{\lambda'} \in S_\ell(\Gamma_0(D), \varepsilon)$ has Fourier coefficients in $\mathcal{O}_{K'}$ where $K' \supseteq K$.

Deligne's construction of $\lambda'$-adic representations:

There exists a $\rho_{\tilde{F}_{\lambda'},\lambda'} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}_{\tilde{F}_{\lambda'}})$ satisfying

$$\begin{aligned} \mathrm{char}(\rho_{\tilde{F}_{\lambda'},\lambda'}(\mathrm{Frob}_p)) &= x^2 - a_p(\tilde{F}_{\lambda'})x + p^{\ell-1}\varepsilon(p) \\ &\equiv x^2 - a_p(F_\lambda)x + \varepsilon(p) \pmod{\lambda'} \\ &\equiv x^2 - a_p(f)x + \varepsilon(p). \end{aligned}$$

Hence $\rho_{\tilde{F}_{\lambda'},\lambda'}$ reduced mod $\lambda'$ has the desired properties; that is,

$$\bar{\rho}_{\tilde{F}_{\lambda'},\lambda'} : G_{\mathbb{Q}} \to \mathrm{GL}_2\left(\mathcal{O}_{K',\lambda'}/\lambda'\right).$$

This reduced representation can be conjugated to take values in $\mathrm{GL}_2(\mathcal{O}_{K,\lambda}/\lambda)$ following an argument similar to the one of this morning. $\square$

$$f \rightsquigarrow \rho_{f,\lambda} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}_{K_f}/\lambda)$$

We have now constructed a mod $\lambda$ representation that has all the properties we want, thus the only remaining problem is:

Problem 1. *Lift these mod $\lambda$ representations to a common representation in characteristic 0.*