Math 726: L-functions and modular forms Fall 2011

Lecture 25 : Applications of Rankin-Selberg

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Before continuing, recall that we stated the following facts:

Proposition 1. *Let f be a normalized newform of weight k, level N, and character χ. Write*

$$f = \sum a_n q^n.$$

Then

1. *The field $K = \mathbb{Q}(a_n : n \in \mathbb{N})$ is a finite extension of $\mathbb{Q}$ and each a_n is an algebraic integer (i.e. $a_n \in \mathcal{O}_K$ for each n).*
2. *If $\sigma : K_f \hookrightarrow \mathbb{C}$ is an embedding then $\sigma(f) := \sum_{n=1}^{\infty} \sigma(a_n)q^n$ is a normalized newform of weight k, level N, and character $\sigma \circ \chi$.*
3. *$S_k(\Gamma_1(N))$ has a $\mathbb{Z}$-basis.*

We shall now prove Part 2. First we recall that we have already mentioned how to prove Part 1.

Proof. We do the case $N = 1$, the general case being just a teardrop more technical. First, we claim (exercise) that $S_k(\mathrm{SL}_2(\mathbb{Z}))$ is a free $\mathbb{T}_\mathbb{C}$ module of rank one where

$$\mathbb{T}_\mathbb{C} = \mathbb{C}[T_n : n \in \mathbb{Z}],$$

Let $f_1, \ldots, f_m$ be normalized eigenforms such that

$$S_k(\mathrm{SL}_2(\mathbb{Z})) \cong \oplus_{i=1}^m \mathbb{C} f_i.$$

For $i = 1, \ldots, m$ let $\varphi_i : \mathbb{T}_\mathbb{Q} \to \mathbb{C}$ be the associated homomorphism. In this case $\cap \ker \varphi_i = \{0\}$ because $\varphi_i(\mathbb{T}_\mathbb{Q}) = K_{f_i} = \mathbb{Q}(a_n(f_i) : n \in \mathbb{N})$ so that the image is a field and hence $\ker \varphi_i$ is a maximal ideal of $\mathbb{T}_\mathbb{Q}$. Now choose a subset $\{i(1), \ldots, i(\ell)\} \subseteq \{1, \ldots, m\}$ such that $\ker \varphi_i \neq \ker \varphi_j$ whenever $i \neq j$. Without loss of generality, we may assume that $i(1) = 1$ and so we have

$$\cap_{j=1}^{\ell} \ker \varphi_{\varphi_{i(j)}},$$

Then by the Chinese remainder theorem,

$$\mathbb{T}_{\mathbb{Q}} \cong \mathbb{T}_{\mathbb{Q}}/\cap \ker \varphi_{i(j)} \cong \prod_j \mathbb{T}_{\mathbb{Q}}/\ker \varphi_{i(j)}$$

$$\prod_j \cong K_{f_{i(j)}}$$

and the first factor is K_f. Now

$$\begin{aligned}\mathbb{T}_{\mathbb{C}} &\cong \mathbb{T}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \\ &\cong (K_f \otimes_{\mathbb{Q}} \mathbb{C}) \times (\prod_{j=2}^{\ell} K_{f_{i(j)}}) \otimes_{\mathbb{Q}} \mathbb{C}\end{aligned}$$

and $K_f \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}^r$. The isomorphism is

$$\alpha \otimes 1 \mapsto (\sigma_1(\alpha), \ldots, \sigma_r(\alpha))$$

where $\{\sigma_1, \ldots, \sigma_r\}$ are all the distinct embeddings into $\mathbb{C}$. For each $i = 1, \ldots, r$, let e_i the standard ith basis vector in $(K_f \otimes_{\mathbb{Q}} \mathbb{C}) \times (\prod_{j=2}^{\ell} K_{f_{i(j)}}) \otimes_{\mathbb{Q}} \mathbb{C}$. Then for $n \in \mathbb{N}$, we have

$$T_n(e_i) = \sigma_i(a_n(f))e_i.$$

Since $S_k(\mathrm{SL}_2(\mathbb{Z})) \cong \mathbb{T}_{\mathbb{C}}$ as $\mathbb{T}_{\mathbb{C}}$ modules, e_i in $\mathbb{T}_{\mathbb{C}}$ corresponds to some nonzero $S_k(\mathrm{SL}_2(\mathbb{Z}))$ and so

$$T_n(g) = \sigma_1(a_n(f))g$$

and thus

$$\sigma_i(f) = \frac{1}{a_1(g)}g$$

for all σ_i is a normalized Hecke eigenform. $\square$

1 The First Application of Rankin-Selberg

Now we return to Application I:

Theorem 1. *For a suitable ℓ, the space $M_k(\mathrm{SL}_2(\mathbb{Z}))$ is spanned by the set $\{T_n E_{k-\ell} E_\ell : n \in \mathbb{N}\}$. In fact $\ell \geq 4$ and $k - \ell \geq 4$ will work.*

Thus the products $E_{k-\ell}E_\ell$ are in some sense as far away from eigenforms as possible.

Proof. Recall that $M_k(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}E_k \oplus S_k(\mathrm{SL}_2(\mathbb{Z}))$ as $\mathbb{T}_\mathbb{C}$ modules where

$$E_k = c_k + q + \sum_{n=2}^{\infty} \sigma_{k-1}(n) q^n$$

where $\sigma_{k-1}(n) \sum_{d|n} d^{k-1}$. Also, a quick calculation shows that

$$E_{k-\ell}E_\ell = \frac{c_{k-\ell}c_\ell}{c_k} E_k + \left(E_{k-\ell}E_\ell - \frac{c_{k-\ell}c_\ell}{c_k} E_k \right)$$

for all k and $c_k \neq 0$. We claim (exercise) that the projection $T_{E_k} : M_k(\mathrm{SL}_2(\mathbb{Z})) \to \mathbb{C}E_k$ is in $\mathbb{T}_\mathbb{C}$ and so $\mathbb{C}E_k \subseteq \mathbb{T}_\mathbb{C} E_{k-\ell}E_\ell$. Now we want to show that $S_k(\mathrm{SL}_2(\mathbb{Z})) \subseteq \mathbb{T}_\mathbb{C}(E_{k-\ell}E_\ell)$.

Suppose for the sake of contradiction that the orthogonal complement $(\mathbb{T}_\mathbb{C}E_{k-\ell}E_\ell)^\perp$ with respect to the Petersson inner product is nonzero. Hence we can choose a normalized Hecke eigenform in $\mathbb{T}_\mathbb{C}(E_{k-\ell}E_\ell)^\perp$.

Let $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ and $g \in M_\ell(\mathrm{SL}_2(\mathbb{Z}))$ be eigenforms and suppose that $k > \ell+2$. Then

$$\begin{aligned} D(f, g, k-1) &= L(f \otimes g, k-1)\zeta(2(k-1) + 2 - k - \ell)^{-1} \\ &= \langle E_{k-\ell} g, f \rangle_k. \end{aligned}$$

Take $g = E_\ell$. Then $\langle E_{k-\ell}E_\ell, f \rangle = L(f \otimes g, k-1)\zeta(k-1)^{-1} \neq 0$ and

$$L(f \otimes g, s) = \prod_p L_p(f \otimes g, s)$$

where[1]

$$L_p(f \otimes g, s) = \left[\prod_{i,j=1}^{2} (1 - \alpha_{p,i}\beta_{p,j}p^{-s} \right]^{-1}$$

and

$$\begin{aligned} g = E_\ell &= c_\ell + q + \sum_{n=2}^{\infty} \sigma_{\ell-1}(n) q^n \\ &= \sum_{n=0}^{\infty} b_n q^n \qquad (b_n = \sigma_{\ell-1}(n)). \end{aligned}$$

Therefore, because $\beta_{p,1} = 1$ and $\beta_{p,2} = p^{\ell-1}$,

$$L_p(f \otimes g, s) = (1 - \alpha_{p,1}p^{-s})(1 - \alpha_{p,2}p^{-s})(1 - \alpha_{p,1}p^{\ell-1-s})(1 - \alpha_{p,2}p^{\ell-1-s}),$$

[1] Recall that the α_* and β_* are the roots of the polynomials $x^2 - b_p x + p^{k-1}$ and $x^2 - a_p x + p^{\ell-1}$ respectively.

and so

$$\begin{aligned}
\langle E_{k-\ell}E_\ell, f\rangle &= L(f\otimes g, k-1)\zeta(k-\ell)^{-1}\\
&= L(f,k-1)L(f,(k-1)-\ell+1)\zeta(k-1)^{-1}.
\end{aligned}$$

Suppose $k \geq 12$, $k-\ell \geq k/2+2$, and $k-1 \geq k/2+2$. Since $L(f,s) = \prod_p (1-a_p p^{-s}+p^{k-1-s})^{-1}$ for $\Re(s) \geq k/2+2$ so $L(f,s)$ is nonzero for $\Re(s) \geq k/2+2$. Thus $\langle E_{k-\ell}E_\ell, f\rangle_k \neq 0$ whenever $k-\ell \geq k/2+2$ and $k-1 \geq k/2+2$. $\square$