Math 726: L-functions and modular forms — Fall 2011

Lecture 24 : Appplications of Rankin-Selberg

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We recall briefly part of the morning and continue.

THEOREM 1. *Let $f = \sum_{n=1}^\infty a_n q^n$ be a normalized newform of weight k, level N and character χ. Then the sum $\sum_{p\nmid N} |a_n|^2 p^{-s}$ converges absolutely for $\Re(s) > k$ and*

$$\sum_{p\nmid N} |a_n|^2 p^{-s} = \log\left(\frac{1}{s-k}\right) + O(1)$$

as $s \to k$.

Proof. Consider

$$\begin{aligned}\log L_N(f\otimes \overline{f}, s) &= -\sum_{p\nmid N}\left[\sum_{i,j=1}^2 \log\left(1 - \alpha_{pi}\overline{\alpha_{pj}}p^{-s}\right)\right] \\ &= \sum_{p\nmid N}\left(\sum_{m=1}^\infty \frac{(\alpha_{p,1}^m + \alpha_{p,2}^m)(\overline{\alpha_{p,1}}^m + \overline{\alpha_{p,2}}^m)}{mp^{ms}}\right).\end{aligned}$$

If we let

$$g_m(s) = \sum_{m=1}^\infty \frac{(\alpha_{p,1}^m + \alpha_{p,2}^m)(\overline{\alpha_{p,1}}^m + \overline{\alpha_{p,2}}^m)}{mp^{ms}}$$

then

$$\log L_N(f\otimes\overline{f}) = \sum_{m=1}^\infty g_m(s).$$

Note that

$$\begin{aligned} g_1(s) &= \sum_{p\nmid N} \frac{|\alpha_{p,1} + \alpha_{p,2}|^2}{p^s} \\ &= \sum_{p\nmid N} \frac{|a_p|^2}{p^s} \\ &\le \sum_{m=1}^\infty g_m(s) \\ &= \log\left(\frac{1}{s-k}\right) \text{ as } s \to k.\end{aligned}$$

Here, we have used that $L_N(f \otimes \overline{f}, s)$ has a simple pole at $s = k$. $\square$

Definition 2. Let $\mathbb{P}$ be the set of primes and let $X \subseteq \mathbb{P}$. Define the superior density of X to be

$$\text{dens.}\sup(X) = \limsup_{s \to 1^+} \left(\frac{\sum_{p \in X} p^{-s}}{\log\left(\frac{1}{s-1}\right)} \right).$$

Recall that

$$\sum p^{-s} = \log\left(\frac{1}{s-1}\right) + O(1)$$

as $s \to 1^+$ so that $\text{dens.}\sup(X) \in [0,1]$.

Note that the image of an Artin representation $G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ has finite image so that the set

$$\{a_p : p \in \mathbb{P}\} \subseteq \mathbb{C}$$

is a finite set.

Proposition 1. *Let f be a normalized newform of weight one, level N, and character χ. Then for each $\eta > 0$ there exists sets $X_\eta, Y_\eta \subseteq \mathbb{C}$ such that $|Y_\eta| < \infty$ and*

$$\text{dens.}\sup(X) \leq \eta$$

and $a_p \in Y_\eta$ if $p \notin X_\eta$.

Before proving this, we will state some facts without proof, which we already did for level $N = 1$ when we introduced algebraic modular forms. The general weight case follows from similar arguments.

Proposition 2. *Let f be a normalized newform of weight k, level N, and character χ. Write*

$$f = \sum a_n q^n.$$

Then

1. *The field $K = \mathbb{Q}(a_n : n \in \mathbb{N})$ is a finite extension of $\mathbb{Q}$ and each a_n is an algebraic integer (i.e. $a_n \in \mathcal{O}_K$ for each n).*
2. *If $\sigma : K_f \hookrightarrow \mathbb{C}$ is an embedding then $\sigma(f) := \sum_{n=1}^{\infty} \sigma(a_n) q^n$ is a normalized newform of weight k, level N, and character $\sigma \circ \chi$.*

3. $S_k(\Gamma_1(N))$ has a $\mathbb{Z}$-basis.

Now we prove Proposition 1

Proof. Fix $c > 0$ let

$$Y(c) := \{\alpha \in \mathcal{O}_{K_f} : |\sigma(\alpha)|^2 \leq c \quad \forall \sigma \in \mathrm{Hom}(K_f, \mathbb{C})\}.$$

Since the image $\mathcal{O}_{K_f} \hookrightarrow K_f \otimes_{\mathbb{Q}} \mathbb{R}$ is discrete and cocompact, the set $Y(c)$ is finite. Define

$$X(c) = \{p \in \mathbb{P} : a_p \notin Y(c)\}.$$

By Theorem 2 of Lecture 23 and Part 2 of Proposition 2 in this lecture,

$$\begin{aligned}\sum_p |a_p|^2 p^{-s} &= \log\left(\frac{1}{s-1}\right) + O(1) \\ &= \sum_p |\sigma(a_p)|^2 p^{-s}\end{aligned}$$

for all embeddings $\sigma : K_f \to \mathbb{C}$, so

$$\sum_{\sigma \in \mathrm{Hom}(K_f,\mathbb{C})} \left(\sum_p |\sigma(a_p)|^2 p^{-s}\right) = [K_f : \mathbb{Q}] \log\left(\frac{1}{s-1}\right) + O(1).$$

Since $\sum_\sigma |\sigma(a_p)|^2 \geq c$ whenever $p \in X(c)$,

$$\begin{aligned}c \sum_{p \in X(c)} |\sigma(a_p)|^2 p^{-s} &\leq \sum_{p \in X(c)} \left(\sum_{\sigma \in \mathrm{Hom}(K_f,\mathbb{C})} |\sigma(a_p)|^2\right) p^{-s} \\ &\leq [K_f : \mathbb{Q}] \log\left(\frac{1}{s-1}\right) + O(1)\end{aligned}$$

and so $\mathrm{dens.\,sup}(X(c)) = [K_f : \mathbb{Q}]/c$. Thus we can select $X_\eta = X(c), Y_\eta = Y(c)$. $\square$

Remark 1. We can also derive the result of today's lecture assuming the Petersson conjecture. By this conjecture, we know that $|a_p| \leq 2p^{\frac{k-1}{2}}$ for all $p \nmid N$. Then

$$\log L_N(f \otimes \overline{f}, s) = \log\left(\frac{1}{s-k}\right) + O(1)$$

and

$$\log L_N(f \otimes \overline{f}, s) = \sum_{p \nmid N} |a_p|^2 p^{-s} + \sum_{m=2}^{\infty} g_m$$

so by the Petersson conjecture $\sum_{m=2}^{\infty} g_m(s)$ converges absolutely at $s = k$ and so

$$\sum_{p \nmid N} |a_p|^2 p^{-s} = \log\left(\frac{1}{s-k}\right) + O(1).$$