Lecture 23 : Applications of Rankin-Selberg

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We shall describe the following two applications of Rankin-Selberg:

I For a suitable ℓ, the set $\{T_n E_{k-\ell} E_\ell : n \in \mathbb{N}\}$ spans $M_k(\mathrm{SL}_2(\mathbb{Z}))$, and

II If $f = \sum a_n q^n$ is a newform of weight k, level N, and character χ then $\sum |a_p|^2 p^{-s}$ converges for $\Re(s) > k$, and

$$\sum |a_p|^2 p^{-s} \leq \log\left(\frac{1}{s-k}\right) + O(1).$$

Before proving II we will first look at the simpler case of $\sum_p p^{-s}$.

Theorem 1. $\sum_p p^{-s} = \log\left(\frac{1}{s-k}\right) + O(1)$.

Proof. Recall

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \qquad \Re(s) > 1$$

and $\zeta(s)$ has a simple pole at $s = 1$. Taking logarithms and using the Taylor expansion for the logarithm gives

$$\log \zeta(s) = \sum_p -\log(1 - p^{-s}) = \sum_p \sum_{n=1}^{\infty} \frac{p^{-ms}}{m}$$
$$= \sum_{i=1}^{\infty} g_m(s)$$

where $g_m(s) = \sum_p \frac{p^{-ms}}{m}$ and the last equality is due to switching the order of summation. Since $\zeta(s)$ has a simple pole at $s = 1$ with a nonzero residue

$$\lim_{s\to 1^+} (s-1)\zeta(s) \neq 0,$$

taking logarithms gives

$$\lim_{s\to 1^+} [\log(s-1) + \log \zeta(s)] = O(1).$$

Rearranging gives

$$\lim_{s\to 1^+} \log \zeta(s) = \log\left(\frac{1}{s-1}\right) + O(1)$$

so as $s \to 1^+$,

$$g_1(s) + \sum_{m=2}^{\infty} g_m(s) = \log\left(\frac{1}{s-1}\right) + O(1).$$

We claim that the summation $\sum_{m=2}^{\infty} g_m(s)$ converges for $s = 1$. Indeed,

$$\begin{aligned}
\sum_{m=2}^{\infty} g_m(s) &= \sum_p \sum_{m=2}^{\infty} \frac{p^{-ms}}{m} \\
&= \sum_p \sum_{m'=2}^{\infty} \left(\frac{p^{-2m's}}{2m} + \frac{p^{-2(m'+1)s}}{2m'+1}\right) \\
&\leq \sum_p \sum_{m'=1}^{\infty} \frac{p^{-2m's}}{m'} \\
&= \log \zeta(2s)
\end{aligned}$$

where the inequality comes from the inequality

$$\frac{p^{-2m's}}{2m} + \frac{p^{-2(m'+1)s}}{2m'+1} \leq 2p^{-2m's}2m'.$$

$\square$

Now we prove the more general theorem. We restate it for convenience.

THEOREM 2. *If $f = \sum a_n q^n$ is a newform of weight k, level N, and character χ then $\sum_{p\nmid N} |a_p|^2 p^{-s}$ converges for $\Re(s) > k$, and*

$$\sum_{p\nmid N} |a_p|^2 p^{-s} \leq \log\left(\frac{1}{s-k}\right) + O(1).$$

Note that in this Lecture, we show the convergence, and in Lecture 24 we will show the estimate.

Proof. We have already seen that

$$|a_n| \leq c n^{k/2}$$

for some real $c > 0$. Thus

$$\sum_{m=1}^{\infty} |a_n| n^{-s}$$

converges for $\Re(s) > k/2 + 1$. Let

$$L_N(f \otimes \overline{f}, s) = \prod_{p \nmid N} L_p(f \otimes \overline{f}, s)$$

where

$$L_p(f \otimes \overline{f}, s) = \prod_{i,j=1}^{2} (1 - \alpha_{p,i} \overline{\alpha_{p,j}} p^{-s})^{-1}.$$

Here the $\alpha_{i,j}$ are the roots of $x^2 = a_p x + \chi(p) p^{k-1}$ and $\overline{f} = \sum \overline{a_n} q^n \in S_k(N, \overline{\chi})$. Define

$$D_N(f, \overline{f}, s) = \sum_{(n,N)=1} \frac{|a_n|^2}{n^s}.$$

Then $L_N(f \otimes \overline{f}, s) = D_N(f, \overline{f}, s) \zeta_N(2s - 2 - 2k)$ where

$$\zeta_N(s) = \prod_{p \nmid N} (1 - p^{-s})^{-1} = \prod_{p} \left(\sum_{n=0}^{\infty} \frac{|a_{p^r}|^2}{p^{rs}} \right)$$

is the partial ζ function. Now let

$$D(f, \overline{f}, s) = \sum_{n \in \mathbb{N}} \frac{|a_n|^2}{n^s}.$$

Recall that for normalized Hecke eigenforms $f, g \in S_k(\mathrm{SL}_2(\mathbb{Z}))$,

$$L(f \otimes g, s) = D(f, g, s) \zeta(2s - 2 - 2k)$$

extends to a meromorphic function and has a unique simple pole at $s = k$ if and only if $\langle \overline{f}, g \rangle \neq 0$. In particular, this means that $D(f, \overline{f}, s) \zeta(2s + 2 - 2k)$ extends to a meromorphic function with a unique simple pole at $s = k$ because $\langle \overline{f}, \overline{f} \rangle \neq 0$.

Note that

$$H(k) = \prod_{p|N} (1 - p^{-2}) \prod_{p|N} (1 - |a_p|^2 p^{-k}) \neq 0$$

by our bound $|a_p| < k/2$. Since $L_N(f \otimes \overline{f}, s)$ has a simple pole, $D(f, \overline{f}, s)$ extends to a meromorphic function holomorphic for $\Re(s) > k + 1$, and hence via the Lemma below, a holomorphic function for $\Re(s) > k$ by looking at the product $L_N(f \otimes \overline{f}), s)$. $\square$

LEMMA 1. *Let Φ be a meromorphic function satisfying $\Phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ for $\Re(s) > \sigma \in \mathbb{R}$ for some fixed σ, $c_n \geq 0$. If Φ is holomorphic at $s = \sigma$ then there exists a $\delta > 0$ such that $\Phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ for $\Re(s) > \sigma - \delta$.*

Although we will not prove this lemma, we shall use this lemma to prove that

$$D(f, \overline{f}, s) = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^s}$$

converges absolutely for $\Re(s) > k$, where $f = \sum a_n q^n$ is our newform of weight k. Indeed, suppose that $\sum \frac{|a_n|^2}{n^s}$ diverges at $s = \sigma$ for some $\sigma \in \mathbb{R}$ with $k < \sigma \leq k+1$. Let

$$r_0 = \sup\left\{ r \in \mathbb{R} : k < r \leq k+1, \sum \frac{|a_n|^2}{n^r} \text{ diverges} \right\}.$$

By definition of the supremum $\sum \frac{|a_n|^2}{n^s}$ converges for $\Re(s) > r_0$. We already know that $D(f, \overline{f}, s)$ is holomorphic at $s = r_0$ because it is holomorphic except for a simple pole at $s = k$ and by hypothesis $r_0 > k$; and so by Lemma 1, there is a $\delta > 0$ such that $\sum \frac{|a_n|^2}{n^r}$ converges for $\Re(s) > r_0 - \delta$, and this is a contradiction!

Note, the Weil conjectures imply that $|a_p| \leq 2p^{\frac{k-1}{2}}$ for $k \geq 2$.