Recall Let $g, f$ be modular forms of weight $k$ and $\ell$, respectively, with their fourier expansion given by

$$g = \sum b_n q^n, \quad f = \sum a_n q^n.$$  

In the last session, we defined

$$D(f, g, s) = \sum \frac{a_s b_s}{n^s} = L(f \otimes g, s) \zeta(2s + 2 - k - \ell)^{-1},$$

and proved

**Theorem (Rankin)**

$$D(f, g, k - 1) = \frac{(4\pi)^{k-1}}{\Gamma(k - 1)} < E_{k-\ell} g, f >_k.$$  

And to understand more general values, we introduced

$$E_r(z, s) = \sum_{(m, n) \in \mathbb{Z}^2} \frac{1}{(mz + n)^r} y^s \frac{y^s}{|mz + n|^{2s}},$$

which converges absolutely (and uniformly on compact sets) when $Re(s) > 1 - r/2$, for all $r \in \mathbb{Z}$. This leads naturally to Hecke’s trick of realising the nonholomorphic Eisenstein series of weight two as the limit of $E_2(z, s)$ as $s$ tends to 0.

Then we proved

**Theorem** For all $s \in \mathbb{C}$ with $Re(s) > 2 - k/2 + \ell/2 = 2 - \frac{k-\ell}{2},$

$$D(f, g, k + s - 1) = (4\pi)^{k+s-1} \Gamma(k + s - 1)^{-1} < E_{k-\ell}(z, s) g, f >_k.$$
Remark. This continues to make sense when \( k \neq \ell + 2 \). In particular, it makes sense when \( k = \ell \). In that case, the formula for \( D(f, g, s) \) involves

\[
E(z, s) = \sum_{(m, n) \in \mathbb{Z}^2} \frac{y^s}{|mz + n|^{2s}},
\]

which converges when \( \text{Re}(s) > 1 \).

Now we define

\[
G(z, s) = \sum_{(m, n) \in \mathbb{Z}^2} \frac{y^s}{|mz + n|^{2s}} = \zeta(2s)E(z, s).
\]

Properties of \( G(z, s) \). For \((z, s) \in \mathcal{H} \times \mathbb{C}^{\text{Re}>1}\)

- \( G \), as a function of \( z \) satisfies

\[
G\left(\frac{az + b}{cz + d}, s\right) = G(z, s)
\]

for all \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \).

So \( G(z, s) \) is a nonholomorphic Eisenstein series of weight 0.

- As a function of \( s \), with \( z \) fixed

\[
G(z, s) = \sum_{(m, n) \in \mathbb{Z}^2} \frac{1}{Q_z^s(m, n)},
\]

where \( Q_z(m, n) = \frac{|mz + n|^2}{y} \) is a quadratic form in two variables, with \( \text{disc}(Q_z) = -4 \).

\text{Proof.} \ Let \( z = x + iy \),

\[
\frac{1}{y}|mz + n|^2 = \frac{1}{y}(mz + n)(m\bar{z} + n) = \frac{1}{y}(m^2z\bar{z} + mn(z + \bar{z}) + n^2)
\]

So,

\[
\Delta = \frac{1}{y^2}[(z + \bar{z})^2 - 4z\bar{z}] = \frac{1}{y^2}(z - \bar{z})^2 = -4
\]

and,

\[
\text{disc}(Q_z) = -4.
\]

\( G(z, s) \) is also called the Epstein zeta function attached to \( Q_z \), when considered as a function of \( s \).
Lemma

\[ < G(z, s)g(z), f(z) >_k = (4\pi)^{1-k-s}\Gamma(k-1+s)L(f \otimes g, k-1+s). \]

Proof.

\[ < G(z, s)g, f >_k = \zeta(2s) < E(z, s)g, f >_k = \zeta(2s)(4\pi)^{1-k-s}\Gamma(k-1+s)L(f \otimes g, k-1+s)\zeta(2(k-1+s) + 2 - 2k)^{-1} = (4\pi)^{1-k-s}\Gamma(k-1+s)L(f \otimes g, k-1+s). \]

\[ \square \]

Theorem Let \( z \in \mathcal{H} \) be fixed,

1. The function \( G(z, s) \) has a meromorphic continuation to \( s \in \mathbb{C} \) and is entire except for a simple pole with residue \( \pi \) at \( s = 1 \).

2. \( G^*(z, s) := \pi^{-s}\Gamma(s)G(z, s) \) is holomorphic except for simple poles at \( s = 1 \) and \( s = 0 \) with residues 1 and -1 respectively. Further,

\[ G^*(z, s) = G^*(z, 1-s). \]

Proof. (Sketch)

Step 1 Let

\[ \Theta_z(t) = \sum_{(m,n) \in \mathbb{Z}^2} e^{-\pi Q_z(m,n)t}, \]
a two variable Guassian. We consider the Mellin transform of \( \Theta_z \) and we get the following formula,

\[ G^*(z, s) = \Gamma(s) \sum_{(m,n) \in \mathbb{Z}^2} (\pi Q_z(m,n))^{-s} = \int_0^\infty (\Theta_z(t) - 1)t^{s} \frac{dt}{t}. \]

Step 2 Poisson summation formula gives

\[ \Theta_z(1/t) = t\Theta_z(t). \]

Step 3

\[ G^*(z, s) = \int_1^\infty (\Theta_z(t) - 1)t^{1-s} \frac{dt}{t} + \int_1^\infty (\Theta_z(t) - 1)t^{s} \frac{dt}{t} + \left[ \frac{1}{s-1} - \frac{1}{s} \right]. \]

\[ \square \]

Reference For a detailed proof see Notes on Modular Forms (of one variable), by D. Zagier.
Integral Representation of $L(f \otimes g, k - 1 + s)$ Define

$$
\Lambda(f \otimes g, k - 1 + s) := < G^*(z, s) g, f >_k \\
= 4^{1-k}(2\pi)^{-2s} \Gamma(s) \Gamma(k - 1 + s)L(f \otimes g, k - 1 + s)
$$

This function has nice symmetries;

**Theorem**

1. $\Lambda(f \otimes g, k - 1 + s)$ extends to a meromorphic function of $s$.
2. It is holomorphic except at $s = 0, 1$ where it has simple poles with residues $- < g, f >$ and $< g, f >$.

**Proof.** This follows directly from the analytic continuation of $G^*(z, s)$.

$$
\text{Res}_{s=0} < G^*(z, s) g(z), f(z) >_k = < \text{Res}_{s=0} G^*(z, s) g(z), f(z) >_k \\
= - < g, f > .
$$

**Corollary**

1. $L(f \otimes g, s)$ extends to a meromorphic function to $s \in \mathbb{C}$.
2. The function has at worst a simple pole at $s = k$.
3. This pole is present if and only if $< f, g > \neq 0$.

**Remark**

- All our calculations were done with modular forms of level 1 and of equal weight.
- It can also be made to work for modular forms of level $N$, by working with linear combinations of $G(dz, s)$ for $d|N$.

Next Week We will discuss two applications of Rankin method;

1. $E_{\ell} E_{k-\ell} \in M_k(SL_2(\mathbb{Z}))$ is as far from being an eigenform as possible, for suitable $\ell$. 

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**Theorem**  For suitable $\ell$, the set $\{T_n(E_{\ell}E_{k-\ell})\}$ spans the space $M_k(SL_2(\mathbb{Z}))$.

2. For $f = g$ use the analytic properties of $L(f \otimes \bar{f}, s)$, in particular, the pole at $s = k$ to infer information about the rates of growth of $|a_n(f)|$

$$D(f, \bar{f}, s) = \sum |a_n|^2 n^{-s}.$$