Math 726: L-functions and modular forms — Fall 2011

Lecture 22: Rankin-Selberg Method, continued

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Recall Let g, f be modular forms of weight k and ℓ, respectively, with their fourier expansion given by

$$g = \sum b_n q^n, \qquad f = \sum a_n q^n.$$

In the last session, we defined

$$\mathcal{D}(f, g, s) = \sum \tfrac{a_n b_n}{n^s} = L(f \otimes g, s)\zeta(2s + 2 - k - l)^{-1},$$

and proved

Theorem (Rankin)

$$\mathcal{D}(f, g, k-1) = \frac{(4\pi)^{k-1}}{\Gamma(k-1)} < E_{k-\ell}g, f >_k .$$

And to understand more general values, we introduced

$$E_r(z, s) = \sum_{(m,n)\in(\mathbb{Z}^2)'} \frac{1}{(mz+n)^r} \frac{y^s}{|mz+n|^{2s}},$$

which converges absolutely (and uniformly on compact sets) when $Re(s) > 1 - r/2$, for all $r \in \mathbb{Z}$. This leads naturally to Hecke's trick of realising the nonholomorphic Eisenstein series of weight two as the limit of $E_2(z, s)$ as s tends to 0.

Then we proved

Theorem For all $s \in \mathbb{C}$ with $Re(s) > 2 - k/2 + \ell/2 = 2 - \frac{k-\ell}{2}$,

$$\mathcal{D}(f, g, k+s-1) = (4\pi)^{k+s-1}\Gamma(k+s-1)^{-1} < E_{k-\ell}(z, s)g, f >_k .$$

Remark This continues to make sense when $k \not> \ell + 2$. In particular, it makes sense when $k = \ell$. In that case, the formula for $\mathcal{D}(f, g, s)$ involves

$$E(z, s) = \sum_{(m,n) \in (\mathbb{Z}^2)'} \frac{y^s}{|mz + n|^{2s}},$$

which converges when $Re(s) > 1$.

Now we define

$$G(z, s) = \sum_{(m,n) \in \mathbb{Z}^2}{}' \frac{y^s}{|mz + n|^{2s}} = \zeta(2s) E(z, s).$$

Properties of $G(z, s)$ For $(z, s) \in \mathcal{H} \times \mathbb{C}^{Re>1}$

- G, as a function of z satisfies

  $$G(\frac{az + b}{cz + d}, s) = G(z, s)$$

  for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$.

  So $G(z, s)$ is a nonholomorphic Eisenstein series of weight 0.

- As a function of s, with z fixed

  $$G(z, s) = \sum_{(m,n) \in \mathbb{Z}^2}{}' \frac{1}{Q_z^s(m, n)},$$

  where $Q_z(m, n) = \frac{|mz+n|^2}{y}$ is a quadratic form in two variables, with $disc(Q_z) = -4$.

Proof. Let $z = x + \imath y$,

$$\begin{aligned} \frac{1}{y}|mz + n|^2 &= \frac{1}{y}(mz + n)(m\bar{z} + n) \\ &= \frac{1}{y}(m^2 z\bar{z} + mn(z + \bar{z}) + n^2) \end{aligned}$$

So,

$$\Delta = \frac{1}{y^2}[(z + \bar{z})^2 - 4z\bar{z}] = \frac{1}{y^2}(z - \bar{z})^2 = -4$$

and,

$$disc(Q_z) = -4.$$

□

$G(z, s)$ is also called the Epstein zeta function attached to Q_z, when considered as a function of s.

Lemma

$$< G(z,s)g(z), f(z) >_k = (4\pi)^{1-k-s}\Gamma(k-1+s)L(f\otimes g, k-1+s).$$

Proof.

$$\begin{aligned} < G(z,s)g, f >_k &= \zeta(2s) < E(z,s)g, f >_k \\ &= \zeta(2s)(4\pi)^{1-k-s}\Gamma(k-1+s)L(f\otimes g, k-1+s)\zeta(2(k-1+s)+2-2k)^{-1} \\ &= (4\pi)^{1-k-s}\Gamma(k-1+s)L(f\otimes g, k-1+s). \end{aligned}$$

□

Theorem Let $z \in \mathcal{H}$ be fixed,

1. The function $G(z,s)$ has a meromorphic continuation to $s \in \mathbb{C}$ and is entire except for a simple pole with residue π at $s = 1$.
2. $G^*(z,s) := \pi^{-s}\Gamma(s)G(z,s)$ is holomorphic except for simple poles at $s = 1$ and $s = 0$ with residues 1 and -1 respectively. Further,

$$G^*(z,s) = G^*(z, 1-s).$$

Proof. (Sketch)

Step 1 Let

$$\Theta_z(t) = \sum_{(m,n)\in\mathbb{Z}^2} e^{-\pi Q_z(m,n)t},$$

a two variable Guassian. We consider the Mellin transform of Θ_z and we get the following formula,

$$G^*(z,s) = \Gamma(s) \sum_{(m,n)\in\mathbb{Z}^2}{}' [\pi Q_z(m,n)]^{-s} = \int_0^\infty (\Theta_z(t) - 1)t^s \frac{dt}{t}.$$

Step 2 Poisson summation formula gives

$$\Theta_z(1/t) = t\Theta_z(t).$$

Step 3

$$G^*(z,s) = \int_1^\infty (\Theta_z(t)-1)t^{1-s}\frac{dt}{t} + \int_1^\infty (\Theta_z(t)-1)t^s\frac{dt}{t} + [\frac{1}{s-1} - \frac{1}{s}].$$

□

Reference For a detailed proof see Notes on Modular Forms (of one variable), by D. Zagier.

Integral Representation of $L(f \otimes g, k-1+s)$ Define

$$\begin{aligned}\Lambda(f \otimes g, k-1+s) &:=< G^*(z,s)g, f >_k \\ &= 4^{1-k}(2\pi)^{-2s}\Gamma(s)\Gamma(k-1+s)L(f \otimes g, k-1+s)\end{aligned}$$

This function has nice symmetries;

Theorem

1. $\Lambda(f \otimes g, k-1+s)$ extends to a meromorphic function of s.
2. It is holomorphic except at $s = 0, 1$ where it has simple poles with residues $- < g, f >$ and $< g, f >$.

Proof. This follows directly from the analytic continuation of $G^*(z,s)$.

$$\begin{aligned}Res_{s=0} < G^*(z,s)g(z), f(z) >_k &=< Res_{s=0}G^*(z,s)g(z), f(z) >_k \\ &= - < g, f > .\end{aligned}$$

□

Corollary

1. $L(f \otimes g, s)$ extends to a meromorphic function to $s \in \mathbb{C}$.
2. The function has at worst a simple pole at $s = k$.
3. This pole is present if and only if $< f, g > \neq 0$.

Remark

- All our calculations were done with modular forms of level 1 and of equal weight.
- It can also be made to work for modular forms of level N, by working with linear combinations of $G(dz, s)$ for $d|N$.

Next Week We will discuss two applications of Rankin method;

1. $E_\ell E_{k-\ell} \in M_k(SL_2(\mathbb{Z}))$ is as far from being an eigenform as possible, for suitable ℓ.

Theorem For suitable ℓ, the set $\{T_n(E_\ell E_{k-\ell})\}$ spans the space $M_k(SL_2(\mathbb{Z}))$.

2. For $f = g$ use the analytic properties of $L(f \otimes \bar{f}, s)$, in particular, the pole at $s = k$ to infer information about the rates of growth of $|a_n(f)|$

$$\mathcal{D}(f, \bar{f}, s) = \sum |a_n|^2 n^{-s}.$$