Math 726: L-functions and modular forms — Fall 2011

Lecture 21: Rankin-Selberg Method, continued

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In the last lecture, we proved that

$$L(f \otimes g, s) = \zeta(2s + 2 - k - \ell)\mathcal{D}(f, g, s),$$

where $\mathcal{D}(f, g, s) = \sum_{n=1}^{\infty} a_n b_n n^{-s}$. Now we will investigate the analytic properties of $\mathcal{D}(f, g, s)$.

Goal To obtain a formula for $\mathcal{D}(f, g, k-1)$ assuming $k > \ell + 2$.

We will be using the following Eisenstein Series in expressing the above value

$$E_{k-\ell}(z) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ gcd(m,n)=1}} \frac{1}{(mz+n)^{k-\ell}}.$$

Theorem

$$\mathcal{D}(f, g, k-1) = \frac{(4\pi)^{k-1}}{\Gamma(k-1)} < E_{k-\ell}g, f >_k .$$

Note $E_{k-\ell}g$ is not an eigenform anymore; Hecke operators respect addition, but not multiplication.

Proof.

$$\begin{aligned} < E_{k-\ell}g, f >_k &= \int_{SL_2(\mathbb{Z})\diagdown\mathcal{H}} y^k E_{k-\ell}(z) g(z) \bar{f}(z) \frac{dxdy}{y^2} \\ &= \int_{SL_2(\mathbb{Z})\diagdown\mathcal{H}} \sum_{(m,n)\in(\mathbb{Z}^2)'} \frac{y^k}{(mz+n)^{k-\ell}} g(z)\bar{f}(z)\frac{dxdy}{y^2}, \end{aligned} \tag{1}$$

where $(\mathbb{Z}^2)' = \{(m,n) \in \mathbb{Z}^2 | gcd(m,n) = 1\}$. To compute (1) we use the following lemma.

Define $U := \{\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}\}$, the subgroup of 2×2 unipotent matrices, then we have;

Lemma

1. The map $U\diagdown SL_2(\mathbb{Z}) \to (\mathbb{Z}^2)'$ defined by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (c, d)$ is a bijection and we have,

2. $\frac{y^k}{(mz+n)^{k-\ell}} g(z)\bar{f}(z) = y(\gamma z)^k g(\gamma z)\bar{f}(\gamma z) \qquad \gamma = \begin{bmatrix} * & * \\ m & n \end{bmatrix}$.

Proof. 1. given $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$, clearly $(c,d) \in (\mathbb{Z}^2)'$, and is left unchanged by multiplying by any matrix of the form $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$. Injectivity and surjectivity are obvious.

2. Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\begin{aligned} y(\gamma z)^k g(\gamma z)\bar{f}(\gamma z) &= \frac{y(z)^k}{(mz+n)^k(\overline{mz+n})^k}(mz+n)^\ell g(z)(\overline{mz+n})^k \bar{f}(z) \\ &= \frac{y^k}{(mz+n)^{k-\ell}} g(z)\bar{f}(z). \end{aligned}$$

□

So we have,

$$\begin{aligned} (1) &= \int_{SL_2(\mathbb{Z})\diagdown\mathcal{H}} \sum_{U\diagdown SL_2(\mathbb{Z})} y(\gamma z)^k g(\gamma z)\bar{f}(\gamma z)\frac{dxdy}{y^2} \\ &= \int_{SL_2(\mathbb{Z})\diagdown\mathcal{H}} \sum_{U\diagdown SL_2(\mathbb{Z})} y(\gamma z)^k g(\gamma z)\bar{f}(\gamma z)\frac{dx(\gamma z)dy(\gamma z)}{y^2(\gamma z)} \\ &= \int_{U\diagdown\mathcal{H}} y^k g(z)\bar{f}(z)\frac{dxdy}{y^2}. \end{aligned} \tag{2}$$

This method is called (Rankin's) unfolding.

Now as $U\diagdown\mathcal{H} = \{x + \imath y | 0 \le x \le 1, y > 0\}$, we have,

$$\begin{aligned} (2) &= \int_{y=0}^{\infty}\int_{x=0}^{1} y^k g(z)\bar{f}(z)\frac{dxdy}{y^2} \\ &= \int_{y=0}^{\infty}\int_{x=0}^{1} y^k (\sum_{n=1}^{\infty} b_n e^{2\pi\imath nz})(\sum_{m=1}^{\infty} \overline{a_m} e^{-2\pi\imath m\bar{z}})\frac{dxdy}{y^2} \\ &= \int_{y=0}^{\infty}\int_{x=0}^{1} y^k \sum_{n,m=1}^{\infty} b_n\overline{a_m} e^{2\pi\imath n(x+\imath y)} e^{-2\pi\imath m(x-\imath y)}\frac{dxdy}{y^2}. \end{aligned} \tag{3}$$

But f is an eigenform for the Hecke operators and in the full level, these operators are self-adjoint, so all the Fourier coefficients are real and we can replace $\overline{a_m}$ with a_m, to get,

$$(3) = \int_0^{\infty} y^k \sum_{n,m=1}^{\infty} b_n a_m e^{-2\pi(n+m)}(\int_0^1 e^{2\pi\imath(n-m)x}dx)\frac{dy}{y^2}. \tag{4}$$

But the innermost parentheses is equal to the Kronecker delta function,

$$
\begin{aligned}
(4) &= \int_{y=0}^{\infty} y^k \sum_{n=1}^{\infty} a_n b_n e^{-4\pi n y} \frac{dy}{y^2} \\
&= \sum_{n=1}^{\infty} a_n b_n \int_{y=0}^{\infty} y^{k-1} e^{-4\pi n y} \frac{dy}{y} \\
&= (\sum_{n=1}^{\infty} \frac{a_n b_n}{(4\pi n)^{k-1}}) \int_0^{\infty} u^{k-1} e^{-u} \frac{du}{u} \\
&= \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \mathcal{D}(f, g, k-1),
\end{aligned}
$$

where we have made the change of variable $u = 4\pi n y$ for each integral in the sum. Note that one can check the convergence of the series $\Sigma_{n=1}^{\infty} \frac{a_n b_n}{n^{k-1}}$, using the estimates for $|a_n|$ and $|b_n|$ and the fact that $k > \ell + 2$. □

Problem Can we get a similar formula for $\mathcal{D}(f, g, k-1+s)$?

Idea Introduce the non-holomorphic Eisenstein series of weight $k - \ell$,

$$
\begin{aligned}
E_{k-\ell}(z.s) &= \sum_{(m,n)\in(\mathbb{Z}^2)'} \frac{1}{(mz+n)^{k-\ell}} \frac{y^s}{|mz+n|^{2s}} \\
&= \sum_{\gamma = \begin{bmatrix} a & b \\ m & n \end{bmatrix} \in U \diagdown SL_2(\mathbb{Z})} \frac{1}{(mz+n)^{k-\ell}} y(\gamma z)^s.
\end{aligned}
$$

Some Properties of $E_{k-\ell}$

- $E_k(z, s)$ converges for $Re(s) >> 0$ (for any k.)
- $E_k(\frac{az+b}{cz+d}, s) = (cz+d)^k E_k(z, s)$, so E_k behaves like a modular form as a function of z, but is not holomorphic in z.
- $E_k(z, s)$ is holomorphic in s and converges for $Re(s) >> 0$.

Theorem For $Re(s) >> 0$,

$$
< E_{k-\ell}(z, s) g, f >_k = \frac{1}{(4\pi)^{k+s-1}} \Gamma(k+s-1) \mathcal{D}(f, g, k+s-1).
$$

note

- The assumption $Re(s) >> 0$ ensures that the sum in $E_{k-\ell}$ and the integral in Peterson product both converge.
- We recover the previous formula by setting $s = 0$.

Proof. The proof is exactly the same as the proof of the previous theorem; One can show

$$< E_{k-\ell}(z,s)g, f >_k = \int_{y=0}^{\infty} \int_{x=0}^{1} y^{k+s} g(z) \bar{f}(z) \frac{dxdy}{y^2},$$

using Ranking's unfolding method, and by similar computations as the previous theorem prove the assertion. □

Goal Use the integral representation of $\mathcal{D}(f, g, s)$ to get an analytic continuation and functional equation for it.

Key Ingredient Functional equation satisfied by $E_k(z, s)$ as a function of s.