Math 726: L-functions and modular forms — Fall 2011

Lecture 20: The Rankin-Selberg Method

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Setting Let
$$f = \sum_{n=1}^{\infty} a_n q^n \in S_k(SL_2(\mathbb{Z})),$$
and
$$g = \sum_{n=1}^{\infty} b_n q^n \in S_\ell(SL_2(\mathbb{Z}))$$
be eigenforms (hence newforms, as we are working in level 1.) Recall that
$$\begin{aligned} L(f,s) &= L(V_f,s) \\ &= \sum_{n=1}^{\infty} a_n n^{-s} \\ &= \prod (1 - a_p p^{-s} + p^{k-1-2s})^{-1} \\ &= \prod (1-\alpha_p p^{-s})^{-1}(1-\alpha'_p p^{-s})^{-1} \end{aligned}$$
is the Hecke L-function corresponding to the compatible system of ℓ-adic representations attached to f. Here $(x-\alpha_p)(x-\alpha'_p) = x^2 - a_p x + p^{k-1}$.

And similarly for g
$$\begin{aligned} L(g,s) &= L(V_g,s) \\ &= \sum_{n=1}^{\infty} b_n n^{-s} \\ &= \prod (1 - b_p p^{-s} + p^{\ell-1-2s})^{-1} \\ &= \prod (1-\beta_p p^{-s})^{-1}(1-\beta'_p p^{-s})^{-1}, \end{aligned}$$
with $(x-\beta_p)(x-\beta'_p) = x^2 - b_p x + p^{\ell-1}$.

Next we define the Rankin L-series and investigate its analytic properties;

Definition The *Rankin L-series*, or *Rankin convolution L-series,* attached to (f,g) is
$$\begin{aligned} L(f\otimes g,s) &:= L(V_f\otimes V_g,s) \\ &= \prod (1-\alpha_p\beta_p p^{-s})^{-1}(1-\alpha_p\beta'_p p^{-s})^{-1}(1-\alpha'_p\beta_p p^{-s})^{-1}(1-\alpha'_p\beta'_p p^{-s})^{-1}, \end{aligned} \tag{1}$$

where $V_f \otimes V_g$, the tensor product of the two representation, is 4 dimensional, so that $L(f \otimes g, s)$ is defined by an Euler product with factors of degree 4.

As we saw before, Hecke Theory implies that $L(f,s)$ has analytic continuation and satisfies a functional equation. We would like to show something similar for the Rankin L-series.

To begin, we want to know if there is a formula describing A_n, where $L(f \otimes g, s) = \sum A_n n^{-s}$. The formula for general n looks complicated a priori, so let us start by calculating A_p, for p a prime number. We have

$$\begin{aligned}(1) = \prod (1 + \alpha_p \beta_p p^{-s} + \alpha_p^2 \beta_p^2 p^{-2s} + \dots)(1 + \alpha_p \beta'_p p^{-s} + \alpha_p^2 {\beta'}_p^2 p^{-2s} + \dots) \\ (1 + \alpha'_p \beta_p p^{-s} + {\alpha'}_p^2 \beta_p^2 p^{-2s} + \dots)(1 + \alpha'_p \beta'_p p^{-s} + {\alpha'}_p^2 {\beta'}_p^2 p^{-2s} + \dots),\end{aligned}$$

so,

$$\begin{aligned}A_p &= \alpha_p \beta_p + \alpha_p \beta'_p + \alpha'_p \beta_p + \alpha'_p \beta'_p \\ &= (\alpha_p + \alpha'_p)(\beta_p + \beta'_p) = a_p b_p.\end{aligned}$$

Yet, in general, A_{p^j} need not equal $a_{p^j} b_{p^j}$. These two facts motivate us to ask how does the Rankin L-series differ from the 'modified Rankin L-series', as a first approximation of the Rankin L-series;

Definition The *Modified Rankin L-series* attached to f and g is the function

$$\mathcal{D}(f,g,s) = \sum_{n=1}^{\infty} a_n b_n n^{-s}.$$

Remark The function $a_n b_n$ is weakly multiplicative, and therefore

$$\mathcal{D}(f,g,s) = \prod_p (1 - a_p b_p p^{-s} + a_{p^2} b_{p^2} p^{-2s} + \dots).$$

In what follows, our analysis will be local, that is, prime by prime.

Lemma Let $(B_{p^j})_{j=1,2,\dots}$ be a sequence of complex numbers satisfying an r-term linear recurrence of the form

$$B_{p^0} = 1$$

and

$$B_{p^{j+r}} = \lambda_1 B_{p^{j+r-1}} + \lambda_2 B_{p^{j+r-2}} + \dots + \lambda_r B_{p^j},$$

for all $j \geq 0$. Then

$$1 + B_p x + B_{p^2} x^2 + \ldots = \frac{Q(x)}{1 - \lambda_1 x - \lambda_2 x^2 - \cdots - \lambda_r x^r}$$

for some $Q(z) \in \mathbb{C}[x]$, of degree strictly less than r.

Proof. Consider

$$(1 + B_p x + B_{p^2} x^2 + \ldots)(1 - \lambda_1 x - \lambda_2 x^2 - \cdots - \lambda_r x^r),$$

and observe that it has no terms of degree $\geq r$, using the recurrence formula. $\square$

Application Take $B_{p^j} = a_{p^j} b_{p^j}$. This sequence satisfies a recurrence of order 4, as we see below.

Lemma The sequence B_{p^j} satisfies a recurrence of the form

$$B_{p^{j+4}} = \lambda_1 B_{p^{j+3}} + \lambda_2 B_{p^{j+2}} + \lambda_3 B_{p^{j+1}} + \lambda_4 B_{p^j},$$

where

$$(1 - \lambda_1 x - \lambda_2 x^2 - \lambda_3 x^3 - \lambda_4 x^4) = (1 - \alpha_p \beta_p x)(1 - \alpha_p \beta'_p x)(1 - \alpha'_p \beta_p x)(1 - \alpha'_p \beta'_p x).$$

Proof. a_{p^j} satisfies a two term recurrence

$$a_{p^{j+2}} = a_p a_{p^{j+1}} - p^{k-1} a_{p^j}.$$

Let W be the vector space of all sequences satisfying this linear recurrence

$$x_{p^{j+2}} = a_p x_{p^{j+1}} - p^{k-1} x_{p^j}.$$

Then we have $dim(W) = 2$ and a basis for this vector space is given by

$$(\alpha_p^j)_{j=1,2,\ldots} \quad and \quad (\alpha'^j_p)_{j=1,2,\ldots}.$$

This can be seen by considering the linear transformation on W which shifts each sequence one term down (so that $(x_1, x_2, \ldots)$ gets mapped to $(x_2, \ldots)$.) This is an invertible map, as the first term of any sequence in W is determined by its second and third term, using the recurrence formula. The eigenvalues of this transformation are geometric progressions, and for a gemetric progression to be in W, i.e. to satisfy the recurrence formula, its ratio should satisfy $x^2 - a_p x + p^{k-1}$, and so is equal to α_p or α'_p.
Hence, $(a_{p^j})_j$ is a linear combination of $(\alpha_p^j)_j$ and $(\alpha'^j_p)_j$. Likewise, $(b_{p^j})_j$ is a linear combination of $(\beta_p^j)_j$ and $(\beta'^j_p)_j$.

Hence $(B_{p^j})_j = (a_{p^j} b_{p^j})_j$ is a linear combination of the four geometric progressions

$$(\alpha_p \beta_p)^j \text{ and } (\alpha_p \beta'_p)^j \text{ and } (\alpha'_p \beta_p)^j \text{ and } (\alpha'_p \beta'_p)^j.$$

And these easily can be seen to satisfy a recurrence of the desired form.

$\square$

Corollary

$$1 + a_p b_p p^{-s} + a_{p^2} b_{p^2} p^{-2s} + \cdots = \frac{Q(p^{-s})}{(1-\alpha_p\beta_p p^{-s})(1-\alpha'_p\beta_p p^{-s})(1-\alpha_p\beta'_p p^{-s})(1-\alpha'_p\beta'_p p^{-s})}$$

for $Q(x) \in \mathbb{C}[x]$, with $deg(Q) \leq 3$.

$\square$

So to understand the relation between $L(f \otimes g, s)$ and $\mathcal{D}(f \otimes g, s)$ it remains to compute Q.

Computation By grouping pairwise terms of the following product we can write

$$\begin{aligned}&(1-\alpha_p\beta_p p^{-s})(1-\alpha'{}_p\beta_p p^{-s})(1-\alpha_p\beta'{}_p p^{-s})(1-\alpha'{}_p\beta'{}_p p^{-s})\\ &= (1 - a_p\beta_p p^{-s} + \beta_p^2 p^{k-1-2s})(1 - a_p\beta'{}_p p^{-s} + \beta'^2_p p^{k-1-2s}),\end{aligned} \tag{2}$$

which is the product of values of the characteristic polynomial of α_p at $\beta_p p^{-s}$ and $\beta'_p p^{-s}$. Now we have

$$\begin{aligned}(2) = 1 &- a_p b_p p^{-s} + [p^{k-1}\beta'^{\,2}_p + a_p^2 p^{\ell-1} + \beta_p{}^2 p^{k-1}]p^{-2s}\\ &- [a_p\beta'_p p^{\ell-1}p^{k-1} + a_p\beta_p p^{\ell-1}p^{k-1}]p^{-3s} + p^{2(k+\ell-2)}p^{-4s}\end{aligned}$$

which, using $\beta_p^2 + \beta'^2_p = b_p^2 - 2p^{\ell-1}$, is equal to

$$(1 - a_p b_p p^{-s}) + [b_p^2 p^{k-1} + a_p^2 p^{\ell-1} - 2p^{k+\ell-2}]p^{-2s} - [a_p b_p p^{k+\ell-2}]p^{-3s} + p^{2(k+\ell-2)}p^{-4s},$$

a polynomial in p^{-s} with coefficients in $\mathbb{Z}[a_p, b_p]$.

Computation 2 Another tedious but rewarding computation shows that

$$\begin{aligned}Q(p^{-s}) &= (1 + a_p b_p p^{-s} + a_{p^2} b_{p^2} p^{-2s} + \dots)\\ &\times (1 - a_p b_p p^{-s} + [b_p^2 p^{k-1} + a_p^2 p^{\ell-1} - 2p^{k+\ell-2}]p^{-2s} - [a_p b_p p^{k+\ell-2}]p^{-3s} + p^{2(k+\ell-2)}p^{-4s})\\ &= 1 - p^{k+\ell-2}p^{-2s}.\end{aligned}$$

Hence we have proved the following theorem;

Theorem

$$\mathcal{D}(f, g, s) = L(f \otimes g, s)\zeta(2s + 2 - k - \ell)^{-1}.$$

It now remains to understand

$$\mathcal{D}(f, g, s) = \sum_{n=1}^{\infty} a_n b_n n^{-s}.$$