Math 726: L-functions and modular forms — Fall 2011

Lecture 2 : Functional equation of the Riemann ζ-function

Instructor: Henri Darmon — *Notes written by: Luca Candelori*

In the first part of this course we will be concerned with analytic properties of L-functions, such as analytic continuations and functional equations. We will start by exploring the analytic properties of the most elementary L-function, the *Riemann ζ function*. This is a function of $s \in \mathbb{C}$ defined by:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

For $\Re[s] > 1$ the above series converges uniformly on compact subsets of $\mathbb{C}$, and therefore it is an analytic function there. The goal of this lecture is to present Riemann's proof of the functional equation of ζ relating $\zeta(s)$ to $\zeta(1-s)$. As a by-product, we will see that $\zeta(s)$ has meromorphic continuation to all of $\mathbb{C}$ with a simple pole at $s = 1$.

The functional equation of ζ is stated in terms of the **Γ-function**, a classical complex analytic function whose basic properties we briefly recall.

Definition 1. For $s \in \mathbb{C}$, the Γ-function is defined as:

$$\Gamma(s) := \int_0^{\infty} e^{-t} t^s \, \frac{dt}{t}.$$

Note that the integral defining the Γ-function converges at ∞ for all s, but at 0 it only converges for $\Re[s] > 0$. How can we then extend Γ to all of $\mathbb{C}$? The idea is to use the following property of $\Gamma(s)$:

Theorem 2 (Functional equation of $\Gamma(s)$). *For all s such that $\Re[s] > 0$,*

$$\Gamma(s+1) = s\Gamma(s)$$

Proof. Exercise. (Hint: use integration by parts.) $\square$

Using this functional equation, we can extend Γ to $\Re[s] < 0$ by recursively setting $\Gamma(s) := \Gamma(s+1)/s$ (note that the pole at 0 is nevertheless carried over in the analytic continuation). Consequently, we obtain

- $\Gamma(s)$ extends to a meromorphic function on all of $\mathbb{C}$ with simple poles at all negative integers.
- For all positive integers n,

$$\Gamma(n) = (n-1)!$$

Therefore Γ can be viewed as a complex analytic function interpolating the values of the factorial function.

We are now ready to state the functional equation of the Riemann zeta function:

THEOREM 3 (Functional equation of $\zeta(s)$). *Let* $\Lambda(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$. *Then*

$$\Lambda(s) = \Lambda(1-s).$$

for all s *with* $\Re[s] > 1$.

Now by definition $\zeta(s)$ converges for $\Re[s] > 1$. Thanks to the functional equation of Theorem 3, we can extend $\zeta(s)$ to $\Re[s] < 0$. Convergence on the remaining strip $0 \leq \Re[s] \leq 1$ (the **critical strip**) will be deduced as a by-product of the proof of Theorem 3.

We will follow Riemann's proof of Theorem 3, which will lend itself to a wide range of generalizations. The proof exploits the **theta function** $\theta : \mathbb{R}_{>0} \to \mathbb{C}$ given by:

$$\theta(t) := \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}.$$

We want to view this function as a **Mellin transform**.

DEFINITION 4. Let $g : \mathbb{R}_{>0} \to \mathbb{C}$ be a continuous function of rapid decay (i.e. $|g(t)| \ll t^{-N}\ \forall N \geq 0$). Then the Mellin transform of g is the function:

$$M(g)(s) := \int_0^\infty g(t)t^s \,\frac{dt}{t}$$

Note that the rapid decay of g implies that the integral defining the Mellin transform always converges at ∞.

EXAMPLE 5. $\Gamma(s) = M(e^{-t})(s)$.

In the proof of Theorem 3 the basic principle is that $\Lambda(s)$ **essentially is the Mellin transform of** θ. The transformation properties of θ (which is our first example of a 'modular form', to be defined later) then translate into the functional equation of Λ via the Mellin transform.

An immediate problem with this idea is that $\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}$ is **not a function of rapid decay**, since the constant term in the series is not of rapid decay. We then replace θ by:

$$\omega(t) := \sum_{n=1}^{\infty} e^{-\pi n^2 t}$$

which is related to θ by:

$$\theta(t) = 1 + 2\omega(t) \quad , \quad \omega(t) = \frac{\theta(t) - 1}{2}.$$

The function $\omega(t)$ is of rapid decay, and therefore we can take its Mellin transform.

THEOREM 6.

$$M(\omega)(s) = \pi^{-s}\Gamma(s)\zeta(2s) = \Lambda(2s)$$

Proof. The proof is a type of computation which we will see again later in the course. By definition, we have:

$$M(\omega)(s) = \int_0^\infty \omega(t) t^s \,\frac{dt}{t} = \int_0^\infty \left(\sum_{n=1}^{\infty} e^{-\pi n^2 t}\right) t^s \,\frac{dt}{t}.$$

Now all the terms in the infinite series are of rapid decay, and therefore we can switch the order of integration (exercise!):

$$\int_0^\infty \left(\sum_{n=1}^{\infty} e^{-\pi n^2 t}\right) t^s \,\frac{dt}{t} = \sum_{n=1}^{\infty} \int_0^\infty e^{-\pi n^2 t} \cdot t^s \,\frac{dt}{t}.$$

Each term of the series looks almost like a Γ function. In fact, if we make the change of variables $u = \pi n^2 t$ for each term in the series, we get:

$$\begin{aligned}
\sum_{n=1}^{\infty} \int_0^\infty e^{-\pi n^2 t} \cdot t^s \,\frac{dt}{t} &= \sum_{n=1}^{\infty} \int_0^\infty e^{-u} \pi^{-s} n^{-2s} u^s \,\frac{du}{u} \\
&= \pi^{-s} \cdot \left(\int_0^\infty e^{-u} u^s \,\frac{du}{u}\right) \cdot \left(\sum_{n=1}^{\infty} n^{-2s}\right) \\
&= \pi^{-s}\Gamma(s)\zeta(2s)
\end{aligned}$$

□

The point of expressing $\Lambda(s)$ as the Mellin transform of $\omega(t)$ is that ω enjoys nice transformation properties coming from those of θ.

THEOREM 7 (Functional equation for θ). *For all $t > 0$,*

$$\theta\left(\frac{1}{t}\right) = \sqrt{t} \cdot \theta(t).$$

Proof. The proof, which uses Poisson summation, will be presented in the next lecture. $\square$

COROLLARY 1 (Functional equation for ω). *For all $t > 0$,*

$$\omega\left(\frac{1}{t}\right) = \sqrt{t} \cdot \omega(t) + \frac{\sqrt{t}}{2} - \frac{1}{2}.$$

Proof.

$$\begin{aligned} \omega\left(\frac{1}{t}\right) &= \frac{\theta(1/t) - 1}{2} \\ &= \frac{\sqrt{t} \cdot \theta(t) - 1}{2} \\ &= \frac{\sqrt{t} \cdot (1 + 2\omega(t)) - 1}{2} \\ &= \sqrt{t} \cdot \omega(t) + \frac{\sqrt{t}}{2} - \frac{1}{2} \end{aligned}$$

$\square$

We are now ready to prove the functional equation of $\zeta(s)$ and its analytic continuation (to the critical strip as well).

Proof of Theorem 3. By Theorem 6 we know that:

$$\Lambda(s) = M(\omega)(s/2) = \int_0^\infty \omega(t) t^{s/2} \, \frac{dt}{t}.$$

This integral converges for all s near ∞, since ω is of rapid decay. However, the convergence at 0 will depend on the growth of $\omega(t)$ near 0. Now

$$\omega(t) \approx C \cdot t^{-1/2} \quad \text{as } t \to 0 \quad \text{(Exercise)}$$

and therefore the integral converges provided $\Re[s] > 1$ (we know that Λ has a pole at $s = 1$ coming from ζ, therefore we cannot hope to go past that just by using the definition).

Next, we break down the integral into two pieces:

$$\int_0^\infty \omega(t) t^{s/2} \, \frac{dt}{t} = \int_0^1 \omega(t) t^{s/2} \, \frac{dt}{t} + \int_1^\infty \omega(t) t^{s/2} \, \frac{dt}{t}. \tag{1}$$

Note that the second integral in (1) converges for all $s \in \mathbb{C}$, whereas the first integral only converges for $\Re[s] > 1$. We would then like to change the first integral into one that looks like the second, i.e. with limits from 1 to ∞ and with ω in the integrand. Of course, this can be accomplished with the substitution $t \to 1/t$ and by using the functional equation for ω:

$$\begin{aligned}\int_0^1 \omega(t) t^{s/2} \frac{dt}{t} &= \int_\infty^1 \omega\left(\frac{1}{t}\right) t^{-s/2} \frac{-dt}{t} \qquad \text{(substitution } t \to 1/t) \\ &= \int_1^\infty \left(\sqrt{t}\cdot\omega(t) + \frac{\sqrt{t}}{2} - \frac{1}{2}\right) t^{-s/2} \frac{dt}{t} \qquad \text{(functional equation of } \omega.)\end{aligned}$$

Substituting into (1) we obtain:

$$\begin{aligned}\Lambda(s) = \int_0^\infty \omega(t) t^{s/2} \frac{dt}{t} &= \int_1^\infty \omega(t) t^{\frac{1-s}{2}} \frac{dt}{t} + \int_1^\infty \omega(t) t^{s/2} \frac{dt}{t} + \frac{1}{2}\int_1^\infty t^{\frac{-1-s}{2}}\, dt - \frac{1}{2}\int_1^\infty t^{-1-s/2}\, dt \\ &= \int_1^\infty \omega(t) t^{\frac{1-s}{2}} \frac{dt}{t} + \int_1^\infty \omega(t) t^{s/2} \frac{dt}{t} - \frac{1}{1-s} - \frac{1}{s}.\end{aligned}$$

From this expression we deduce that $\Lambda(s)$ has meromorphic continuation to all of $\mathbb{C}$ with simple poles at $s = 0$ and $s = 1$ and moreover that

$$\Lambda(s) = \Lambda(1-s).$$

□

Since $\Lambda(s) = \pi^{s/2}\Gamma(s/2)\zeta(s)$, we obtain the following consequences of Theorem 3:

- <u>$\zeta(s)$ has a pole at $s = 1$</u> (since $\Gamma(s)$ is analytic at $s = 1/2$ but $\Lambda(s)$ has a simple pole at $s = 1$).
- <u>$\zeta(s)$ is analytic at 0 and $\zeta(0) = -1/2 \neq 0$</u> (since $\Gamma(s)$ has a simple pole at $s = 0$ and so does $\Lambda(s)$).
- <u>$\zeta(s)$ vanishes at all even integers < 0</u> (since $\Gamma(s)$ has poles at negative integers but $\Lambda(s)$ does not).

These observations should give a rough picture of how $\zeta(s)$ looks like in the region $\Re[s] < 1$.