# Lecture 15 : Algebraic Modular Forms

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Recall the following theorem:

Theorem 1. *For all normalised Hecke newforms $f \in S_k(\Gamma_0(N), \chi)$, with Fourier coefficients in a number field $K$, there exists a compatible system $\{V_{f,\lambda}\}_{\lambda \in Spec(\mathcal{O}_K)}$ of $\lambda$-adic $K$-rational representations of $G_{\mathbb{Q}}$ such that*

$$L(\{V_{f,\lambda}\}_{\lambda \in Spec(\mathcal{O}_K)}, s) = L(f, s).$$

*More precisely,*

- *For all primes $p \nmid N\ell$ (where $\ell = Norm_{\mathbb{Q}}^K(\lambda)$), $V_{f,\lambda}$ is unramified at $p$, and*

$$det((1 - xFrob_p)|_{V_{f,\lambda}}) = 1 - a_p(f)x + \chi(p)p^{k-1}x^2;$$

- *For all primes $p \mid N$, $V_{f,\lambda}$ is ramified at $p$ (even when $\lambda \nmid p$), and*

$$det((1 - xFrob_p)|_{V_{f,\lambda}^{I_p}}) = 1 - a_p(f)x.$$

The **main goal** of the next few lectures is now to give some idea of how $\{V_{f,\lambda}\}$ is constructed from $f$.

1. (*Weight of $f$ is 2*) The representations $\{V_{f,\lambda}\}$ are obtained from étale cohomology. Consider the congruence subgroup $\Gamma_1(N)$. If we take the quotient $Y_1(N) := \Gamma_1(N)\backslash\mathcal{H}$ of the upper half-plane by the action of this group, we get a Riemann surface, whose compactification is usually denoted $X_1(N)$ (geometrically, $X_1(N)$ is obtained from $Y_1(N)$ by adding the cusps). A priori, $X_1(N)$ is a curve defined over $\mathbb{C}$, but one can show that it actually admits a *natural model* over the field $\mathbb{Q}$ of rational numbers, that is, $X_1(N)$ can actually be defined over $\mathbb{Q}$. The representations we are looking for are then realised as the action of $G_{\mathbb{Q}}$ on the cohomology groups $H^1_{ét}(X_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$. Luckily, in this case, these groups have a nice interpretation, namely $H^1_{ét}(X_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ is the dual of $V_\ell(J_1(N)_{\overline{\mathbb{Q}}}) := \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim J_1(N)[\ell^n](\overline{\mathbb{Q}})$, where $J_1(N)$ is the *Jacobian* of the modular curve $X_1(N)$, $J_1(N)[\ell^n]$ is the group of $\ell^n$-torsion points of $J_1(N)$, and where the inverse limit is taken with respect to the usual $\ell$-power maps. This construction was first done by Eichler and Shimura (in the case of primes with good reduction), and then Igusa completed it by identifying and analysing the behaviour at the primes of bad reduction.

2. (*Weight of $f$ is strictly greater than 2*) Recall that $Y_1(N)$ is a *moduli space* of isomorphism classes of complex elliptic curves and $N$-torsion data. In this case, we have a *universal elliptic curve* $\mathcal{E}$ over $Y_1(N)$

$$\begin{array}{c} \mathcal{E} \\ \downarrow \\ Y_1(N) \end{array}$$

which is also called a *Kuga-Sato surface*. Generalizing this situation, Deligne considered the fibered product (over $Y_1(N)$) of $k-2$ copies of $\mathcal{E}$:

$$\begin{array}{c} \mathcal{E}^{k-2} \\ \downarrow \\ Y_1(N) \end{array}$$

which is called the *open Kuga-Sato variety* of dimension $k-1$. Let $\mathcal{W}_{k-2}(N)$ denote the compactification of $\mathcal{E}^{k-2}$. Then, the representations$\{V_{f,\lambda}\}$ occur in the cohomology groups $H^{k-1}_{\acute{e}t}(\mathcal{W}_{k-2}(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$.

Note that from the Weil conjectures (more specifically, the *Riemann hypothesis for varieties over finite fields*), we can deduce that $\{V_{f,\lambda}\}$ is of weight $k-1$, that is, the eigenvalues of $Frob_p$ are of complex absolute value $p^{\frac{k-1}{2}}$.

EXAMPLE 2. Consider the cusp form $\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24} \in S_{12}(SL_2(\mathbb{Z}))$, whose $q$-expansion is given by $\Delta = \sum_{n=1}^{\infty}\tau(n)q^n$, where $\tau(n)$ is the *Ramanujan $\tau$-function*. In this case, $\tau(p)$ is the trace of $Frob_p$ acting on $V_{\Delta,\lambda}$, which gives the estimate $|\tau(p)| < 2p^{\frac{11}{2}}$, the so-called *Ramanujan conjecture* (compare this estimate with the "elementary" estimate $|\tau(p)| < Cp^6$).

3. (*Weight of $f$ is 1*) For modular forms of weight one, there is no direct cohomological construction of $\{V_{f,\lambda}\}$. In this case, Serre and Deligne related Hecke eigenforms $f \in S_1(\Gamma_0(N), \chi)$ to (complex) Artin representations, reducing this case to the result for higher weight. This reduction, which does not require any algebraic geometry but rather some useful analytic estimates on Fourier coefficients of cusp forms, will be explained in detail in this class.

We would now like to give an **algebro-geometric interpretation of modular forms and modular curves**; the starting point will be the lattices in $\mathbb{C}$. The main tool we will use is *Weierstrass theory* of complex elliptic curves, which gives a bijection

$$\left\{ \begin{array}{c} \text{Lattices} \\ \Lambda \text{ in } \mathbb{C} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Pairs } (E,\omega) \\ E \text{ is a complex elliptic curve} \\ \omega \text{ is a generator of } \Omega^1(E/\mathbb{C}) \end{array} \right\},$$

where, to a lattice $\Lambda \subset \mathbb{C}$, we attach the pair $(\mathbb{C}/\Lambda, dz)$, and to a pair $(E,\omega)$, we attach the lattice

$$\Lambda_{E,\omega} := \left\{ \int_\gamma \omega \mid \gamma \in H_1(E(\mathbb{C}), \mathbb{Z}) \right\}.$$

Using this bijection, we can now view classical modular form as functions on the set of pairs $(E,\omega)$ defined over $\mathbb{C}$. We are thus led to the following definition.

Definition 3. A *marked elliptic curve* over a field $k$ is a pair $(E,\omega)$ consisting of an elliptic curve $E$ over $k$, together with a $k$-vector space generator $\omega$ of $\Omega^1(E/k)$.

Actually, one can even define marked elliptic curves over any ring $R$. For our purposes, this will simply be a Weierstrass equation with coefficients in $R$, like

$$E : y^2 = x^3 + ax + b, a, b \in R,$$

where we require that the discriminant $\Delta(E)$ be a unit in $R$.

Definition 4. A *weakly homolomorphic algebraic modular form* $f$ of weight $k$ and level 1 over a ring $S$ is a rule, which to every pair $(E,\omega)_R$ consisting of a marked elliptic curve over an $S$-algebra $R$ assigns an element $f(E,\omega)_R \in R$, subject to the following conditions:

- (*Compatibility with base change*) For all homomorphisms $\phi : R \to R'$ of $S$-algebras

$$\phi(f(E,\omega)_R) = f((E,\omega) \otimes_\phi R');$$

- For all $\lambda \in R^\times$, $f(E, \lambda\omega)_R = \lambda^{-k} f(E,\omega)_R$.

We now give some important examples of elliptic curves over rings.

Example 5.

1. **The "universal" elliptic curve over $\mathbb{C}$**

   For this example, we let

   - $R$= ring of holomorphic functions on $\mathcal{H} = \{\tau \in \mathbb{C} \mid Im(\tau) > 0\}$;
   - $E = \mathbb{C}/\langle 1, \tau\rangle$;
   - $\omega = dz$.

Weierstrass theory constructs from this data two doubly periodic functions with poles at $z = 0$, namely

$$\begin{aligned} x &= \wp_{\langle 1,\tau\rangle}(z) = \frac{1}{z^2} + {\sum_{\lambda\in\Lambda}}' \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right); \\ y &= \wp'_{\langle 1,\tau\rangle}(z) = \frac{-2}{z^3} + {\sum_{\lambda\in\Lambda}}' \frac{-2}{(z-\lambda)^3}. \end{aligned}$$

Moreover, these two functions satisfy the following equation

$$y^2 = x^3 - g_4(\tau) - g_6(\tau),$$

where

$$\begin{aligned} g_4(\tau) &= 60 {\sum_{(m,n)\in\mathbb{Z}^2}}' (m\tau+n)^{-4} = \frac{4\pi^3}{3}\left[1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right] \\ g_6(\tau) &= 140 {\sum_{(m,n)\in\mathbb{Z}^2}}' (m\tau+n)^{-6} = \frac{8\pi^6}{27}\left[1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right]. \end{aligned}$$

Furthermore, we have $\omega = \frac{dx}{y}$. The data $(\mathbb{C}/\langle 1,\tau\rangle, dz)$ is thus a marked elliptic curve over the ring $R$.

2. **The Tate curve over $\mathbb{C}$**

In the above construction, we got a torus by identifying opposite sides of the (possibly tilted) rectangle given by the basis $\{1,\tau\}$. By making some change of variables, namely $t = e^{2\pi i z}$, $q = e^{2\pi i\tau}$, we can give a similar construction by identifying the inner ring and the outer ring of an annulus. More concretely, we define:

- $R$= ring of holomorphic functions on the punctured open disk $D^\times = \{q \in \mathbb{C}^\times \mid |q| < 1\}$;
- $E = \mathbb{C}^\times/\langle q\rangle$;
- $\omega = \frac{dt}{t} = 2\pi i dz$.

Now, after choosing our new coordinate functions $X, Y$ to be

$$y = 2\pi^3 iY, \qquad x = \pi^2 X,$$

we get the following equation for the *Tate curve*:

$$Y^2 = X^3 - \frac{1}{3}\left(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right)X + \frac{3}{27}\left(1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right).$$

Also, we get $\omega = \frac{dx}{y}$. In this case, the *discriminant* is equal to

$$\Delta(E) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \in R^{\times}.$$

REMARK 1. These computations reveal that the Tate curve, which was a priori defined over the ring $R$, can also be viewed as an elliptic curve over the ring $\mathbb{Z}[\frac{1}{6}]((q))$.