Math 726: L-functions and modular forms — Fall 2011

Lecture 14 : l-adic Galois representations

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Let $f \in S_k(\Gamma_0(N), \chi)$ be a Hecke eigenform. In previous lectures, we saw that the L-function of such a modular form has a nice product expansion, namely,

$$L(f,s) = \prod_{p \nmid N}(1 - a_p(f)p^{-s} + \chi(p)p^{k-1-2s})^{-1}\prod_{p|N}(1-a_p(f)p^{-s})^{-1},$$

where $a_p(f)$ is the p^{th} Fourier coefficient of f. The first factors are polynomials of degree two evaluated at p^{-s}, and the last factors, linear polynomials. We are thus led to a natural question:

Question 1. What is the connection between the Hecke L-functions and L-functions attached to two-dimensional representations of $G_{\mathbb{Q}}$, the absolute Galois group of $\mathbb{Q}$?

Before answering this question, let us look at an instructive example.

Example 2. Let $f = E_k = c_0 + q + \sum_{n=2}^\infty \sigma_{k-1}(n)q^n$, where, as usual, $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. Then E_k is a Hecke eigenform of level 1, and so its L-function has the following product expansion:

$$\begin{aligned}
L(E_k, s) &= \sum_{n=1}^\infty \sigma_{k-1}(n)n^{-s} \\
&= \prod_p (1 - \sigma_{k-1}(p)p^{-s} + p^{k-1-2s})^{-1} \\
&= \prod_p (1 - (1+p^{k-1})p^{-s} + p^{k-1-2s})^{-1} \\
&= \prod_p (1 - p^{-s} + p^{k-1-s} + p^{k-1-2s})^{-1} \\
&= \prod_p (1-p^{-s})^{-1}(1+p^{k-1-s})^{-1} \\
&= \zeta(s)\zeta(s-k+1),
\end{aligned}$$

where $\zeta(s)$ is the Riemann ζ-function. If $L(E_k, s)$ were to correspond to the L-function of an Artin representation, then the first factor, namely $\zeta(s)$, corresponds to the trivial representation. However, if the second factor, namely $\zeta(s-k+1)$, was the L-function attached to some Galois representation representation $(\rho^?, V)$, then the image of the Frobenius element at the prime p would have to be

$$\rho^?(Frob_p) = p^{k-1}.$$

In particular, $\rho^?$ cannot be a one-dimensional Artin representation. If it were the case, its image would lie in the roots of unity in $\mathbb{C}^\times$, but $\rho^?(Frob_p)$ clearly has infinite order. In order to define $\rho^?$, we need to include Galois representations with ℓ-adic rather than complex coefficients.

First, we fix a prime ℓ. Let μ_{ℓ^n} denote the set of ℓ^n-th roots of unity in $\overline{\mathbb{Q}}$. Here is a **key remark**: for a prime $p \neq \ell$, the Frobenius element $Frob_p$ acts on μ_{ℓ^n} by sending an element ζ to its p^{th} power ζ^p. That is, once you identitify μ_{ℓ^n} with the cyclic group $\mathbb{Z}/\ell^n\mathbb{Z}$, it acts as multiplication by p modulo ℓ^n.

We thus define $\mathbb{Z}_\ell(1) = \varprojlim \mu_{\ell^n}$, where the map $\mu_{\ell^{n+1}} \to \mu_{\ell^n}$ is the usual ℓ-power map. Note that $\mathbb{Z}_\ell(1)$ is a free $\mathbb{Z}_\ell$-module isomorphic to $\mathbb{Z}_\ell$ (via an identification of μ_{ℓ^n} with $\mathbb{Z}/\ell^n\mathbb{Z}$), and that $G_\mathbb{Q}$ acts continuously on $\mathbb{Z}_\ell(1)$. Moreover, $Frob_p$ acts by multplication by p, for all $p \neq \ell$. We then convert $\mathbb{Z}_\ell(1)$ into a $\mathbb{Q}_\ell$-vector space by defining $\mathbb{Q}_\ell(1) := \mathbb{Z}_\ell(1) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, and we set $\mathbb{Q}_\ell(r) := \mathbb{Q}_\ell(1)^{\otimes r}$. The Galois group $G_\mathbb{Q}$ thus acts naturally on $\mathbb{Q}_\ell(r)$. We denote the corresponding representation by

$$\rho_{\ell,r} : G_\mathbb{Q} \to \mathbb{Q}_\ell^\times.$$

These representations satisfy the following properties:

Property 1.

1. (Rationality) Whenever $p \neq \ell$, $\rho_{\ell,r}(Frob_p) = p^r \in \mathbb{Z}$. Note however that not all elements of G_ℓ map into $\mathbb{Z}$.

2. $\rho_{\ell,r}$ is part of a *compatible system* of Galois representations with coefficients in $\mathbb{Q}_\ell$, as ℓ varies. Namely, for all pair of primes $\ell_1 \neq \ell_2$ non of which is equal to p , we have

$$\rho_{\ell_1,r}(Frob_p) = \rho_{\ell_2,r}(Frob_p).$$

   Hence, these representations agree on Frobenius elements, but in spite of this, the representations $\rho_{\ell_1,r}$ and $\rho_{\ell_2,r}$ are different: $\rho_{\ell_i,r}$ factors through $\mathbb{Q}(\mu_{\ell_i^\infty})$, which are linearly disjoint.

3. (Weight property) The representation $\rho_{\ell,r}$ has *weight* $2r$, which means that $\rho_{\ell,r}(Frob_p)$ has eigenvalue p^r of absolute value $|p^r| = p^r$.

With this simple but prototypical example in mind, we make the following definitions.

Definition 3.

- An *ℓ-adic representation* of $G_{\mathbb{Q}}$ is a finite dimensional $\mathbb{Q}_\ell$-vector space V_ℓ equipped with a continuous homomorphism
$$\rho_{V_\ell} : G_{\mathbb{Q}} \to Aut_{\mathbb{Q}_\ell}(V_\ell).$$
- The representation V_ℓ is said to be *rational* if there exists a finite set S of primes such that V_ℓ is unramified outside of $S \cup \{\ell\}$ (that is, for all primes $p \notin S \cup \{\ell\}$, the action of ρ_{V_ℓ} on the inertia group I_p is trivial), and furthermore, for all $p \notin S \cup \{\ell\}$, the characteristic polynomial of $\rho_{V_\ell}(Frob_p)$ has integral coefficients.
- Finally, the representation ρ_{V_ℓ} is said to be *of weight* j if all the eigenvalues of $\rho_{V_\ell}(Frob_p)$ (when viewed as complex numbers) have absolute value $p^{\frac{j}{2}}$.

We are now ready to define the required generalisation of Artin representations. From now on, let $\mathbb{P}$ be the set of all primes.

Definition 4. A *compatible system* of rational ℓ-adic representations is a collection $\{V_\ell\}_{\ell \in \mathbb{P}}$ of rational ℓ-adic representations such that for all primes $p \neq \ell$ outside a finite set of primes S independent of ℓ, the characteristic polynomial of $\rho_{V_\ell}(Frob_p)$ depends only on p and not on ℓ.

We can also ask for a **slightly stronger requirement** on the representations V_ℓ. Namely, for the primes $p \in S$ (called the *bad primes*), we can still consider the action of $\rho_{V_\ell}(Frob_p)$ on $V_\ell^{I_p}$. One can then ask that the characteristic polynomial
$$L_p(V_\ell, x) = det((1 - x\rho_{V_\ell}(Frob_p))|_{V_\ell^{I_p}})$$
should belong to $\mathbb{Z}[X]$, and should be independent of $\ell \neq p$.

These definitions may seem a little contrived at first, but by way of justification:

1. The representations $\{\mathbb{Q}_\ell(r)\}_{\ell \in \mathbb{P}}$ constructed earlier form a compatible system of ℓ-adic representations of $G_{\mathbb{Q}}$. Note that in this case, the set S may be taken to be empty.
2. We can attach to a compatible system $\{V_\ell\}_{\ell \in \mathbb{P}}$ an L-function, by setting
$$L(\{V_\ell\}_{\ell \in \mathbb{P}}, s) = \prod_{p \in \mathbb{P}} L_p(V_\ell, p^{-s})^{-1},$$
which is a function of the complex variable s. If $\{V_\ell\}_{\ell \in \mathbb{P}}$ is of weight j, then this product converges on the right half-plane $Re(s) > 1 + \frac{j}{2}$. In particular, we get $\zeta(s - k + 1) = L(\{\mathbb{Q}_\ell(r)\}_{\ell \in \mathbb{P}}, s)$.

3. Compatible systems are abundant in nature. More precisely, let W be a smooth projective variety over $\mathbb{Q}$, and consider the étale cohomology groups $\{H^i_{ét}(W_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)\}_{\ell \in \mathbb{P}}$; these are in fact $\mathbb{Q}_\ell$-vector spaces. The functoriality of étale cohomology then implies that $H^i_{ét}(W_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)\}$ inherits an action of $G_\mathbb{Q}$, and these families are actually compatible systems of ℓ-adic representations of $G_\mathbb{Q}$. Moreover, Deligne's proof of the Weil conjectures implies that the weight is $\frac{i}{2}$. In this case, the set S of bad primes corresponds to the set of primes where W has *bad reduction*.

We are now ready to ask a more precise question than Question 1.

QUESTION 5. Are Hecke eigenforms attached to compatible systems of two-dimensional representations of $G_\mathbb{Q}$?

The anwser to this question is *yes*, after a slight extension of the notion of compatible systems of ℓ-adic representation, which we now describe. For any number field K (that is, a finite extension of $\mathbb{Q}$), we define a *compatible system* of K-rational representations as follows: this is a collection $\{V_\lambda\}_{\lambda \in Spec(\mathcal{O}_K)}$ of K_λ-vector spaces (where K_λ is the completion of K at the prime λ) such that

$$\rho_{V_\lambda} : G_\mathbb{Q} \to Aut_{K_\lambda}(V_\lambda)$$

is continuous and K-rational for all $\lambda \in Spec(\mathcal{O}_K)$, and such that this family is a compatible system in the obvious sense.

THEOREM 6. *Let $f \in S_k(\Gamma_0, \chi)$ be a (normalised) Hecke eigenform with Fourier coefficients in K (recall that the Fourier coefficients of f are all algebraic numbers lying in some totally real finite extension of $\mathbb{Q}$). Then there exists a compatible system $\{V_{f,\lambda}\}_{\lambda \in Spec(\mathcal{O}_K)}$ of λ-adic K-rational representations of $G_\mathbb{Q}$ such that*

$$L(\{V_{f,\lambda}\}_{\lambda \in Spec(\mathcal{O}_K)}, s) = L(f, s).$$

This theorem is due to various people:

- **Eichler and Shimura** in the case of weight 2 and completed by **Igusa**, who gave the precise description of the Euler factors at the bad primes;
- **Deligne** in the case of a general weight $k > 2$;
- **Deligne and Serre** in the case of weight 1.

In the next few lectures, we are going to discuss these results, and also try to give some idea of the constructions involved, especially in the weight 2 case.