Math 726: L-functions and modular forms — Fall 2011

Lecture 13 : Hecke Operators and Hecke theory

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We previously proved the following statements about modular forms and Hecke operators in the context of $\Gamma = SL_2(\mathbb{Z})$

1. $M(\Gamma) = \bigoplus_k M_k(\Gamma) = \mathbb{C}[E_4, E_6]$
   which implies in particular that each M_k is finite dimensional and :
   $$M(\Gamma, \mathbb{Q}) = \bigoplus_k M_k(\Gamma, \mathbb{Q}) = \mathbb{Q}[E_4, E_6]$$
   where $M_k(\Gamma, \mathbb{Q})$ is the space of modular forms with rational Fourier coefficients.

2. $M_k(\Gamma)$ has extra structures :
   • Hermitian inner product
   • Hecke Operators $(T_n)_{n\geq 1}$

3. $$M_k(\Gamma) = \bigoplus_{\phi \in Hom_{Alg}(\mathbb{T}, \mathbb{C})} M_k(\Gamma)^{\phi}$$
   where $\mathbb{T} = \mathbb{C}[T_1, T_2, ...] \subset End_{\mathbb{C}}(M_k(\Gamma))$ and $dim_{\mathbb{C}}(M_k(\Gamma))^{\phi} = 1$
   $$= \bigoplus_{\phi} \mathbb{C} f_{\phi}$$
   where f_{ϕ} is the normalized eigenform attached to ϕ, completely characterized by :
   - $T_n(f_{\phi}) = \phi(T_n) f_{\phi}$
   - $a_1(f_{\phi}) = 1$

4. If $f \in S_k(\Gamma)$, $f = \sum_{n=1}^{\infty} a_n q^n$, $L(f, s) = \sum_{n=1}^{\infty} a_n n^{-s}$ then $L(f, s)$ has a functional equation relating s and $k - s$.
   Moreover, if f is a normalized eigenform then :
   $$L(f, s) = \prod_p (1 - a_p p^{-s} + p^{k-1-2s})^{-1}$$
   In particular, $L(f, s) \neq 0$ when $\Re e(s) > 1 + k/2$

What about other congruence groups ?

We will discuss the four statements above in the case of $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$

1. We still have $M(\Gamma) = \bigoplus_k M_k(\Gamma)$ and the $M_k(\Gamma)$'s are still finite dimensional but they need not be generated by Eisenstein Series.

   In particular, this is still the case that

   $$M(\Gamma, \mathbb{Q}) = \bigoplus_k M_k(\Gamma, \mathbb{Q})$$

   and

   $$M_k(\Gamma) = M_k(\Gamma, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

   One can produce elements of $M_k(\Gamma)$ using the following tricks :

   - Any $f \in M_k(SL_2(\mathbb{Z}))$ belongs to $M_k(\Gamma)$
   - If $f \in M_k(SL_2(\mathbb{Z}))$ then $f(dz) \in M_k(\Gamma_0(d))$ (Exercise)
   - $E_l(d_1, z)E_m(d_2, z) \in M_{l+m}(\Gamma_0(lcm(d_1, d_2)))$
   - Hecke translates

   In fact, one can prove that if $\Gamma = \Gamma_0(N)$ and k is large enough, then these basic tricks are enough to generate all of $M_k(\Gamma_0(N))$

2. $M_k(\Gamma)$ is still a Hilbert space with Hecke operators but we have to consider different cases :

   - If n is prime to N,

     $$T_n f(z) = n^{k-1} \sum_{\gamma \in \Gamma_1(N) \backslash M_n(N)} f|_\gamma(z)$$

     where $M_n(N)$ are upper triangular unipotent matrices modulo N of determinant n and

     $$T_n f(z) = n^{k-1} \sum_{\gamma \in \Gamma_0(N) \backslash M_n(N)} f|_\gamma(z)$$

     where $M_n(N)$ are upper triangular matrices modulo N of determinant n
   - If l is prime and $l \nmid N$,

     $$T_l f(q) = \sum_{n=1}^{\infty} a_n l q^n + l^{k-1} \sum_{n=1}^{\infty} a_n < l > f q^{nl}$$

     where $f \in M_k(\Gamma_0(N))$ and $< l >$ is the Diamond Operator defined below.

     These are called **good Hecke Operators**.

- If $l|N$, we still define some kind of Hecke Operators :

$$T_l f(q) = \sum_{n=1}^{\infty} a_{nl} q^n$$

These are called **bad Hecke Operators**.

- **Diamond Operators** $< a >$ for $a \in (\mathbb{Z}/N\mathbb{Z})^*$:

$$< a > f(z) := f|_k \gamma_a(z)$$

where $\gamma_a \in \Gamma_0(N)$ and $\gamma_a \equiv \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix}$ mod(N)

Remark 1. T_l is not self adjoint in general

$$T_l^* =< l > T_l \ (\ if \ l \nmid N)$$

Remark 2. The bad Hecke operators T_l do not commute with their adjoints.

Let $\mathbb{T} = \mathbb{C}(T_l, < a >)$ where $l \nmid N$ and $(a, N) = 1$.
Then $\mathbb{T} \subset End_{\mathbb{C}}(M_k(\Gamma))$ and $\mathbb{T}$ is an algebra of commuting operators. We have :

$$M_k(\Gamma) = \bigoplus_{\phi \in Hom_{Alg}(\mathbb{T}, \mathbb{C})} M_k(\Gamma)^{\phi}$$

but $M_k(\Gamma)^{\phi}$ is not necessary 1-dimensional.
Note that if f_1, $f_2 \in M_k(\Gamma)^{\phi}$ are both normalized then

$$\phi(T_n) = a_n(f_1) = a_n(f_2) \quad (\forall (n, N) = 1)$$

Example 1. Construction of 2 such functions :

Let $f_1 \in S_k(\Gamma_1(M))$ be normalized, with $M|N$, $M \neq N$ and define :

$$\begin{aligned} f_2(z) &= f_1(z) + \lambda f_1(dz) \\ &= \sum_{n=1}^{\infty} a_n q^n + \lambda \sum_{n=1}^{\infty} a_n q^{nd} \end{aligned}$$

where $d|\frac{M}{N}$

This example motivates the following definition :

Definition 2. A modular form in $M_k(\Gamma_1(N))$ which is a linear combination of forms of type

$$g(dz)$$

where $g \in M_k(\Gamma_1(M))$ with $M|N$, $M \neq N$ and $d|\frac{M}{N}$ is called an **old form**.

Considering the space generated by old forms, we have :

Definition 3. $S_k(\Gamma_1(N))^{old} =$ **Space of old forms**
$S_k(\Gamma_1(N))^{new} = (S_k(\Gamma_1(N))^{old})^{\perp} =$ **Space of new forms**
where the orthogonality is relative to the Petersson inner product.

Theorem 4. *(Atkin-Lehner)*
The space $S_k(\Gamma_1(N))^{new}$ decomposes as :

$$S_k(\Gamma_1(N))^{new} = \bigoplus_{\phi \in Hom_{Alg}(\mathbb{T}^{new},\mathbb{C})} S_k(\Gamma_1(N))^{\phi}$$

where $dim_{\mathbb{C}} S_k(\Gamma_1(N))^{\phi} = 1$ and $\mathbb{T}^{new} = \mathbb{T}|_{S_k(\Gamma_1(N))^{new}}$

Definition 5. A normalized eigenform $f \in S_k(\Gamma_1(N))^{new}$ is also called a **new form of weight k and level N**

It remains to discuss $L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s}$ in this context.
Can we find a functional equation ?
We note that there is an extra symmetry on $S_k(\Gamma_1(N))$:

$$Fricke\ or\ Atkin-Lehner\ involution\ :\ \boldsymbol{w}_N \leftrightarrow \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$$

Fact : $\boldsymbol{w}_N$ normalizes $\Gamma_0(N)$ and $\Gamma_1(N)$:

$$\boldsymbol{w}_N \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \boldsymbol{w}_N^{-1} = \begin{pmatrix} d & -c \\ -Nb & a \end{pmatrix}$$

Remark 3. $\boldsymbol{w}_N$ does not commute in general with the action of $\mathbb{T}$:

$$\boldsymbol{w}_N T_l = <l> T_l \boldsymbol{w}_N \quad (\forall l \nmid N)$$

But if $\Gamma = \Gamma_0(N)$ then $\boldsymbol{w}_N$ does commute.
In particular, $\boldsymbol{w}_N f = wf$ for $w \in \{\pm 1\}$

THEOREM 6. *Let* $f \in S_k(\Gamma_0(N))$ *and* $\boldsymbol{w}_N f = wf$ *with* $w \in \{\pm 1\}$ *then :*

$$\begin{aligned} \Lambda(f,s) :&= (2\pi)^{-1} N^{s/2} \Gamma(s) L(f,s) \\ &= (-1)^{k/2} w \Lambda(f, k-s) \end{aligned}$$

proof: Assignment 2 □