Math 726: L-functions and modular forms Fall 2011

Lecture 12 : Hecke Operators and Hecke theory

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Recall that we found a formula for multiplication of operators :

$$T_n T_m = T_{nm} \ when \ gcd(m,n) = 1$$

$$T_{p^r} = T_p T_{p^{r-1}} - p^{k-1} T_{p^{r-2}}$$

In particular, we note that all Hecke Operator commute.

Let $\mathbb{T}_k$ be the algebra generated over $\mathbb{C}$ by the Hecke Operator $(T_1, T_2, ...)$ in $End_{\mathbb{C}}(M_k(SL_2(\mathbb{Z})))$, ie $\mathbb{T}_k \subseteq M_d(\mathbb{C})$ where $d = dim_{\mathbb{C}}(M_k(SL_2(\mathbb{Z})))$

Definition 1. A modular form $f \in M_k(SL_2(\mathbb{Z}))$ is called an **eigenform** or **Hecke eigenform** if it is a simultaneous eigenvector for all the Hecke Operators:

$$T_n(f) = \lambda_n f \quad (\forall \lambda_n \in \mathbb{C} \ \ with f \neq 0)$$

Also, the eigenform f is said to be **normalized** if $a_1(f) = 1$

Remark 1. If f is a normalized eigenform we have :

$$T_n(f) = \lambda_n f \ for \ some \ \lambda_n$$

but

$$a_n(f) = a_1(T_n(f)) = a_1(\lambda_n f) = \lambda_n a_1(f) = \lambda_n$$

since $a_1(f) = 1$.

Hence we can recover the Fourier coefficients of a normalized eigenform by considering the associated eigenvalues of Hecke Operators : $\lambda_n = a_n(f)$.

This leads to the following property :

Multiplicity one property : An eigenspace for any collection $\{\lambda_n\}$ of eigenvalues for T_n is exactly 1-dimensional, spanned by the normalized eigenform.

Theorem 2. *(Hecke) If f is a normalized eigenform on $SL_2(\mathbb{Z})$ and if $L(f,s)$ is its associated L-function then for $\Re e(s) > k/2+1$*

$$L(f,s) = \prod_p (1 - a_p p^{-s} + p^{k-1-2s})^{-1}$$

proof : Relations among the T_n's and T_{p^r}'s imply that if f is a normalized eigenform :

$$a(mn) = a(m)a(n)\ if\ (m,n) = 1$$

and

$$a(p^r) = a(p)a(p^{r-1}) - p^{k-1}a(p^{r-2})$$

thus

$$\begin{aligned} L(f,s) &= \sum_{n=1}^{\infty} a_n n^{-s} \\ &= \prod_p \big(\sum_{r=0}^{\infty} a_p r p^{-rs}\big) \\ &= \prod_p (1 - a(p)p^{-s} + p^{k-1-2s})^{-1} \end{aligned}$$

□

We now see some examples of eigenforms and Dirichlet Series :

- Eisenstein Series : $G_k \in M_k(SL_2(\mathbb{Z}))$ is an eigenform.

Normalizing appropriately, we get $\widetilde{G_k}$ with $a_1(\widetilde{G_k}) = 1$
Now recall that the Fourier expansion of G_k involves $\delta_{k-1}(n) = \sum_{d|n} d^{k-1}$, we have :

$$a_n(\widetilde{G_k}) = \delta_{k-1}(n) = \sum_{d|n} d^{k-1}$$

where δ_{k-1} is multiplicative and satisfy :

$$\delta_{k-1}(p^r) = \delta_{k-1}(p)\delta_{k-1}(p^{r-1}) - p^{k-1}\delta_{k-1}(p^{r-2})$$

- A cuspidal example : $\Delta \in S_{12}(SL_2(\mathbb{Z}))$ is also an eigenform.

Writing the q-exansion of Δ:

$$\Delta(q) = \sum_{n=1}^{\infty} \tau(n) q^n$$

where $\tau(n)$ is the n^{th} Fourier coefficient, also called the Ramanujan τ-function, is in particular multiplicative.

$$= q \prod_n (1-q^n)^{24}$$

Then

$$L(\Delta, s) = \sum_n \tau(n) n^{-s} = \prod_p (1 - \tau(p) p^{-s} + p^{11-2s})^{-1}$$

Now, we can ask how often do we find eigenforms if we look at $S_k(SL_2(\mathbb{Z}))$ when $k > 2$ is even.

THEOREM 3. *(Hecke) The space $M_k(SL_2(\mathbb{Z}))$ has a basis consisting of normalized eigenforms*

$$M_k(SL_2(\mathbb{Z})) = \bigoplus_{\phi \in Hom_{Alg}(\mathbb{T}_k, \mathbb{C})} M_k(SL_2(\mathbb{Z}))^{T_n = \phi(T_n)}$$

where $M_k(SL_2(\mathbb{Z}))^{T_n=\phi(T_n)} := \{f \in M_k(SL_2(\mathbb{Z})) | T_n(f) = \phi(T_n)(f)\}$

proof : $S_k(SL_2(\mathbb{Z}))$ is naturally equipped with a hermitian inner product, **the Petersson scalar product**

$$< f, g >= \int_{SL_2(\mathbb{Z})/\mathcal{H}} y^k f(z) \overline{g(z)} \, \frac{dz\overline{dz}}{y^2}$$

which is non-degenerate and positive.

Key fact : The operator T_n is self adjoint with respect to this pairing:

$$< T_n(f), g >=< f, T_n(g) >$$

By diagonalisibility of self adjoint operators we have :

$$S_k(SL_2(\mathbb{Z})) = \bigoplus_{\phi \in Hom_{Alg}(\mathbb{T}_k, \mathbb{C})} S_k(SL_2(\mathbb{Z}))^{T_n=\phi(T_n)}$$

which is an orthogonal direct sum.
But since we know that $M_k(SL_2(\mathbb{Z}))$ is a direct sum of the line spanned by the Eisenstein Series E_k with the space of cusp forms, we also have such a decomposition for $M_k(SL_2(\mathbb{Z}))$:

$$M_k(SL_2(\mathbb{Z})) = \mathbb{C}.E_k \bigoplus S_k(SL_2(\mathbb{Z}))$$

Note however that $< f, g >$ does not converge for $f, g \in M_k(SL_2(\mathbb{Z}))$, hence is it not a orthogonal direct product. □

PROPOSITION 1. *If $f \in S_k(SL_2(\mathbb{Z}))$ is a normalized eigenform then the Fourier coefficients $a_n(f)$ are algebraic numbers.*
More precisely, $\mathbb{Q}(a_1(f), ..., a_n(f), ...)$ is a totally real extension of $\mathbb{Q}$ of degree at most the dimension of $S_k(SL_2(\mathbb{Z}))$.

proof:

As we saw before, $S_k(SL_2(\mathbb{Z}))$ has a basis consisting of modular forms with rational Fourier coefficients.
Let $(f_1, ..., f_d)$ be such a basis.
T_n preserves the rational vector space $\mathbb{Q}f_1 \bigoplus \mathbb{Q}f_2 \bigoplus .... \bigoplus \mathbb{Q}f_d$ and therefore can be viewed as a d by d matrix with rational entries relative to this basis.
But the eigenvalues of such a matrix are algebraic, and moreover, if we let $\mathbb{T}_k(\mathbb{Q})$ be the $\mathbb{Q}$-algebra generated by $T_2, T_3, T_4, ...$ acting on $S_k(SL_2(\mathbb{Z}))$, then $\mathbb{T}_k(\mathbb{Q})$ is isomorphic to a commutative subring of $M_d(\mathbb{Q})$.
Therefore, any $\phi \in Hom(\mathbb{T}_k(\mathbb{Q}), \overline{\mathbb{Q}})$ has its image contained in an extension of $\mathbb{Q}$ of degree less or equal to $d = dim\ S_k(SL_2(\mathbb{Z}))$
Note : $Hom(\mathbb{T}_k(\mathbb{Q}))$ is totally real by self adjointness.□

REMARK 2. The first interesting case, where $dim(S_k(SL_2(\mathbb{Z}))) > 1$ is $k = 24$

As a possible exercise, one can compute the field of definition of the Fourier coefficients of eigenforms in $S_{24}(SL_2(\mathbb{Z})))$

This complete the study of Hecke Theory and L-functions of modular forms in case of level 1. It remains to study the theory when the level is arbitrary, and discuss the features that generalize and those that do not.