Math 726: L-functions and modular forms — Fall 2011

Week 3, lecture 10: Dimension of spaces of modular forms

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Recall

We are working with cusp forms $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$. For each such f, we defined

$$\Lambda(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s),$$

which satisfies

$$\Lambda(f,s) = (-1)^{k-2}\Lambda(f,k-s).$$

We also defined, for each $k \in \mathbb{Z}_{>2}$ even, the Eisenstein series of weight k

$$E_k(q) \in \mathbb{Q} \otimes (\mathbb{Z}[[q]]).$$

Finally we defined the Ramanujan Δ-function by

$$\Delta = \frac{E_4^3 - E_6^2}{1728} = q + \cdots \in S_{12}(\mathrm{SL}_2(\mathbb{Z})).$$

Dimension of some spaces of modular forms

Before we start proving the theorem stated last time, we will present the following fact.

Fact 1. *The Ramanujan Δ-function satisfies*

$$\Delta(z) \neq 0 \quad \text{for all} z \in \mathcal{H}.$$

There are at least two ways to prove this fact. We will present the idea behind them.

Proof. (1. Using a product expansion for Δ) This proof uses the fact that

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

Call the right-hand side Δ_2. To show that $\Delta_2 = \Delta$, one first shows that $\Delta_2 \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$. By the definition of Δ_2, it is easy to see it satisfies the growth condition at ∞. It suffices then to show that $(\Delta_2)|_k\gamma = \Delta_2$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. One further reduces to the case of showing this equality for $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, which is proved by taking the logarithmic derivative of Δ_2 and comparing it with the Eisenstein series E_2:

$$2\pi i \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n \right) = \text{`}E_2(q)\text{'}.$$

where $\sigma(n) = \sum_{d>0 \ , \ d|n} d$.

One then realizes that Δ/Δ_2 is an element of $M_0(\mathrm{SL}_2(\mathbb{Z}))$, which, as we shall see in the next theorem, is 1-dimensional.

(Cf. section VII.4.4 in 'A Course in Arithmetic' by J.-P. Serre) □

Proof. (2. Using a relation between Δ *and the theory of elliptic curves)* This proof uses the fact that

$$\Delta(\tau) = \mathrm{disc}(\mathbb{C} \,/ < 1, \tau > \ , \ 2\pi i dz).$$

As a consequence, since $\mathbb{C}\,/\langle 1, \tau \rangle$ is non-singular, $\Delta(\tau) \neq 0$.

(Cf. corollary 1.4.2 in 'A First Course in Modular Forms' by Diamond & Shurman.) □

Let us now restate the theorem we mentioned last class. In what follows, we will denote $\Gamma := \mathrm{SL}_2(\mathbb{Z})$.

THEOREM 1. *The spaces* $S_k := S_k(\mathrm{SL}_2(\mathbb{Z}))$ *and* $M_k := M_k(\mathrm{SL}_2(\mathbb{Z}))$ *are finite dimensional. More precisely:*

(1) When $k = 0$ *we have*

$$\dim S_0 = 0 \quad \textit{and} \quad \dim M_0 = 1.$$

(2) If $k < 0$*, then*

$$\dim S_k = \dim M_k = 0.$$

(3) If k *is odd, then*

$$\dim S_k = \dim M_k = 0.$$

(4) When $k = 2$ *we have*

$$\dim M_2 = 0.$$

(5) For any k*,*

$$\dim S_{k+12} = \dim M_k.$$

(6) For $k > 2$ *even,*

$$\dim M_k = 1 + \dim S_k.$$

Proof. *(1)* The quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ has a complex structure which makes it a non-compact Riemann surface.

We also have

$$q \longleftrightarrow \text{ complex structure of } \mathrm{SL}_2(\mathbb{Z})\backslash(\mathcal{H}\cup\mathbb{P}_1(\mathbb{Q})) =: X.$$

Using this complex structure, we can identify M_0 with the space of holomorphic functions on $X(\mathbb{C})$. This implies that

$$M_0 = \mathbb{C}\,.$$

Under the same identification, S_0 correspond to the space of holomorphic functions on $X(\mathbb{C})$ which vanish at the cusp. This implies that

$$S_0 = \{0\}.$$

(2) Suppose $g \in M_{-r}$ for some $r > 0$. Then $g^{12}\Delta^r \in S_0 = \{0\}$. So $g = 0$.

(3) This follows from the fact that

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

(4) The space M_2 can be identified with

$$M_2 = \Omega^1_{X-\{\infty\}}(\log(\infty)),$$

which is the space of differentials on $X - \{\infty\}$ which are holomorphic on $X - \{\infty\}$ and have at worse a simple pole at ∞. This identification is given by

$$f \longleftrightarrow 2\pi i f(z)dz = f(q)\frac{dq}{q}.$$

(Notice that $\frac{dq}{q} = 2\pi i dz$.)

But, by the residue theorem, there can be no residue. So

$$M_2 = \Omega^1_X.$$

Now, since

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H} \cong \mathbb{A}_1(\mathbb{C}),$$

we obtain that

$$\mathrm{SL}_2(\mathbb{Z})\backslash(\mathcal{H}\cup\mathbb{P}_1(\mathbb{Q})) \cong \mathbb{P}_1(\mathbb{C}),$$

which has genus 0 and, hence, $\Omega^1_X = \{0\}$.

Exercise 1. *Prove this in a more 'hands-on' way.*

(5) Note that multiplication by Δ gives an injective linear transformation

$$M_k \longrightarrow S_{k+12}.$$

This map is also surjective because if $f \in S_{k+12}$, then f/Δ is still holomorphic on $\mathcal{H}$ and satisfies the growth condition at ∞.

(6) We have an injective map

$$M_k/S_k \longrightarrow \mathbb{C}$$

defined by

$$f \longmapsto a_0(f),$$

where $f = \sum_{n=0}^{\infty} a_n q^n$ is its Fourier expansion. This implies that

$$\dim\left(M_k/S_k\right) \leq 1.$$

This inequality, together with the existence of the Eisenstein series (which are not cusp forms), yields

$$\dim M_k = 1 + \dim S_k.$$

□

As a consequence of this theorem, we may write the following table.

k	$\dim S_k$	$\dim M_k$
< 0	0	0
0	0	1
2	0	0
4	0	1
6	0	1
8	0	1
10	0	1
12	1	2
14	0	1
16	1	2

Note that combining items (5) and (6) we obtain

$$\dim M_k \sim \frac{k}{12}.$$

Corollary 1. *Consider the graded ring (under multiplication)* $\bigoplus_{k=0}^{\infty} M_k(\mathrm{SL}_2(\mathbb{Z}))$*. It is generated by* E_4 *and* E_6*. In other words,*

$$\bigoplus_{k=0}^{\infty} M_k(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6].$$

Corollary 2. *The complex vector space $M_k(\mathrm{SL}_2(\mathbb{Z}))$ has a basis consisting of modular forms with rational (even integers) Fourier coefficients.*

Hecke Operators

A modular form can be viewed in many different, more structured ways.

Definition 2. A *lattice* in $\mathbb{C}$ is a free $\mathbb{Z}$-module Λ in $\mathbb{C}$ of rank two such that $\mathbb{C}/\Lambda$ is compact (equivalently, $\Lambda = \mathbb{Z}\,\mu_1 \oplus \mathbb{Z}\,\mu_2$, where $\mu_1, \mu_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$).

Let Π denote the space of lattices in $\mathbb{C}$.

Definition 3. A complex-valued function F on Π is said to be *homogeneous* of weight k if

$$F(\lambda\Lambda) = \lambda^{-k} F(\Lambda)$$

for all $\lambda \in \mathbb{C}^\times$ and $\Lambda \in \Pi$. Such an F is called *holomorphic* if

$$\tau \in \mathcal{H} \longmapsto F(\mathbb{Z} \oplus \mathbb{Z}\,\tau)$$

is a holomorphic function.