Euler and Lagrange on Pell's Equation

Joshua Aaron McGill ID: 260352168

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Introduction

The field of number theory is notorious for yielding immensely difficult problems that are deceptively easy to state. One such problem, known as Pell's Equation, was studied by some of the greatest mathematicians in history and was not fully solved until the 18^{th} century.[1] Here we will show early attempts at solving this equation by the Indian Mathematician Brahmagupta ($598 - 668$ CE) who forged an identity still used in modern mathematics. Bhaskara II ($1114 - 1185$ CE), used this identity to develope the Chakravala method which he then used as an efficient means of computing solutions to Pell's Equation.

Nearly 600 years later, Leonard Euler ($1707 - 1783$ CE) gave an in depth analysis of Pell's Equation and other Diophantine equations in his book *Elements of Algebra (1765).* This method leaned on earlier attempts by Wallis and is in some ways reminiscent of the cyclic Chakravala method presented by Bhaskara II. The difference lies in the fact that Euler was far more rigorous in his description and used the principle of induction and a precursor to "infinite descent," whereas Bhaskara II simply stated his results-we are never told how he discovered his ingenious method. Six years after *Elements of Algebra* was published, a thirty-five-year-old Joseph Lagrange ($1736 - 1813$ CE) made significant additions to this book in a volume entitled *Additions.* As well as providing some corrections and comments, Lagrange includes a proof of the reducibility of other quadratic diophantine equations to the Pell Equation. Here we will look deeply at two of the chapters of Euler's book that directly deal with the Pell Equation as well as the additions to those chapters as written by Lagrange. We will see Euler's general method as well as several examples with echoes of Greek mathematics. We will see the Method of Continued Fractions in *The Additions* by Lagrange, to conclude we will inspect how he uses this method to correct Euler's.

Pell's Equation

The statement of the problem is as follows:

1. Let $x, y, a \in \mathbb{I}$, can one find all triples (x, y, a) such that $ax^2 \pm 1 = y^2$?
2. Given any a in the above equation can one always find a suitable x and y?
3. Is it possible to enumerate all such solutions? (this is a typical question in the study of differential equations)

1 Elements of Algebra: Part II, Chapter 6

Notes On Euler In 1765 Euler published his book *The Elements of Algebra*. Today, this book might be considered more in the realm of number theory and analysis, but therein lies the power of this book: Euler attempts to coalesce most of the fundamental mathematics of the time into one volume. In contrast to many modern textbooks, the pace of this book is quite gentle–examples are done in detail, few steps are skipped in proofs, and Euler's questions yield conclusions as nicely as his conclusions yield questions. This pedagogic skill is quite evident in the fact that he has very clear notation, uses words consistently (this can be seen more in the original in part II chapters 6 and 7 during which he discusses "Integer Valued functions of the form $ax^2 + 1$ where the answer is a square" (Chapter 6) and a general solution method to such problems (Chapter 7). In the material presented below I attempted to remain true to Euler's notation and method as much as possible. In my recount of Chapter 7, I have changed the order slightly and have added an initial description of his general procedure.

[1] Note that general "solutions" were found far earlier but as we will see, they involved methods that were required, but never proven, to terminate until Lagrange

1.1 Preliminary steps

Euler begins by assuming a solution to the equation $ax^2 + b = y^2$ and then seeks to find another solution given the first as $af^2 + b = g^2$ and then finds that the resulting second solution is:

$$x = \frac{2gpq - (ap^2 + q^2)f}{ap^2 - q^2} \quad y = \frac{g(ap^2 + q^2) - 2afpq}{ap^2 - q^2}$$

which is of the simplified form: $x = ng - mf, y = mg - naf$ and considers the squares of n and m:

$$m^2 = \frac{a^2p^2 + 2ap^2q^2 + q^4}{a^2p^2 - 2ap^2q^2 + q^4} \text{ and } n^2 = \frac{4p^2q^2}{a^2p^2 - 2ap^2q^2 + q^4}$$

This subsequently implies:

$$m^2 - an^2 = \frac{a^2p^2 + 2ap^2q^2 + q^4 - 4ap^2q^2}{a^2p^2 - 2ap^2q^2 + q^4} = \frac{a^2p^2 - 2ap^2q^2 + q^4}{a^2p^2 - 2ap^2q^2 + q^4} = 1$$

Thus m and n satisfy the equation: $m^2 = an^2 + 1$, which he mistakenly attributes to Pell (pg: 343, 344), more on that later. Thus in order to find the solution to $ax^2 + b = y^2$ one must first solve the corresponding Pell's equation. He also uses this to imply that once we have have found one (f, g) we can find infinitely many other solutions to the equation.

1.2 Triangular, Square, and Pentagonal Numbers

Historical Note In our discussion of Greek Mathematics we noted the existence of square and triangular numbers. We should note here that there also exist pentagonal, hexagonal and n-gon numbers defined in a like manner. Here Euler attempts to find numbers that fit into two or more of these classes–a problem that was also of great interest to the Greeks. In class, we considered the case where four times a triangular number summed with unity is a square number. This equation can be written as $4(\frac{x(x+1)}{2} + 1 = y^2 \implies 2x^2 + 2x + 1 = y^2$. This also happens to be the first problem Euler deals with in this section:

Example 1.1. Question 2: Triangular Square numbers The case of all triangular numbers that are also square numbers is handled as follows: α is a triangular number if it satisfies the equation $\alpha = \frac{x(x+1)}{2}$ for some $x \in \mathbb{N}$. β a square number if it (obviously) satisfies the equation $\beta = y^2$ for some $y \in \mathbb{N}$. So in order to find all numbers that are both triangular and square we take:

$$\alpha = \beta \implies \frac{x^2 + x}{2} = y^2$$

Next Euler Multiplies by 8 and adds one to each side to get: $4x^2 + 4x + 1 = (2x + 1)^2 = 8y^2 + 1$ which is is an example of the pell equation:

$$z^2 = 8y^2 + 1 \text{ where } z = 2x + 1 \implies x = \frac{z - 1}{2}$$

0 and 1 are obvious solutions according to Euler, so are 1 and 3.

Example 1.2. Question 3: Square Pentagonal Numbers

The case of all square numbers that are also square numbers is handled as follows: Let z be a root then a pentagon will be of the form $\frac{3z^2 - z}{2}$, and a square will be of the form x^2, so we will have the equation $3z^2 - z = 2x^2$; then we complete the square to get: $36z^2 - 12z + 1 = (6z - 1)^2 = 24x^2 + 1$. Now we set $24x^1 + 1 = y^2$, d conclude that $y = 6z - 1$ thus $z = \frac{y+1}{6}$. Thus we have a Pell Equation with $a = 24, b = 1$.

Next we see that the case of $f = 0$ yields $g = 1$;

and as we must have $24n^2 + 1 = m^2$, we shall make $n = 1$ which gives $m = 5$; so that we shall have $x = 5f + g$ and $y = 5g + 24f$ and not only $z = \frac{y+1}{6}$, but also $z = \frac{1-y}{6}$, because we may write $y = 1 - 6z$

He then lists the first trivial results. He also solves the problem of all numbers that are both triangular and pentagonal. As well as all hexagonal numbers that are also squares. To do this he describes how to deal with the middle term in an equation of the form $ax^2 + bx + c = y^2$ given an initial solution $af^2 + bf + c = g^2$

1.3 The method of Removing a Linear term from a quadratic Integer Equation

An equation of the form $ax^2_bx + c = y^2$ is problematic because we have thus far not discussed methods for dealing with quadratics with linear terms. Euler shows that we may "expunge" this term given enough manipulation. The process begins as follows: Let $ax^2 + bx + c = y^2$ be an equation of integer values with initial solution $af^2 + bf + c = g^2$. Subtract the second from the first to get $a(x^2 - f^2) + b(x - f) = y^2 - g^2$. Now factor:

$$a(x - f)(x + f) + b(x - f) = (y - g)(y + g) = (x - f)(ax + af + b)$$

Multiply both sides by pq to get $pq(x - f)(ax + af + b) = (y - g)(y + g)pq$ by which Euler arrives at

1. $p(x - f) = q(y - p)$
2. $q(ax + af + b) = p(y + g)$

Now multiply the first expression by p and the second expression by q, then subtract the first from the second. Now regroup to get:

$$(aq^2 - p^2)x + (aq^2 + p^2)f + bq^2 = 2gpq \implies x = \frac{2gpq - 2afq^2 - bq^2}{aq^2 - p^2}$$

Substitute this x into $p(x - f) = q(y - g)$

$$p(x - f) = p\frac{2gpq - (aq^2 + p^2)f - bq^2}{aq^2 - p^2} = q(y - g)$$

Now we divide both sides by q and multiply both sides by p to get:

$$\frac{p^2}{q} \cdot \frac{2gpq - 2afq^2 - bq^2}{aq^2 - p^2} = p(y - g) \implies \frac{2gp^2 - 2afpq - bqp}{aq^2 - p^2} = (y - g)$$

$$\implies \frac{2gp^2 - 2afpq - bqp + g(aq^2 - p^2)}{aq^2 - p^2} = y = \frac{g(aq^2 + p^2) - 2afpq - bqp}{aq^2 - p^2}$$

So we are left with:

$$x = \frac{2gpq - 2afq^2 - bq^2}{aq^2 - p^2} \text{ and } y = \frac{g(aq^2 + p^2) - 2afpq - bqp}{aq^2 - p^2}$$

Now we must attempt to get rid of these fractions (because then we can guarantee integer results). To do this Euler notices that if we take p and q to satisfy a Pell equation: $p^2 = aq^2 + 1$, we notice that this is equivalent to $aq^2 - p^2 = -1$ which is exactly our denominator in the case of both x and y. Thus we are left with:

$$x = -2gpq + f(aq^2 + p^2) + bq^2, \text{ and } y = -g(aq^2 + p^2) + 2afpq + bpq$$

But since we began with $af^2 + bf + c = g^2$ and for this g can be either positive or negative, the sign in front of g does not matter. Thus without loss of generality we eliminate the negative sign as follows:

$$x = 2gpq + f(aq^2 + p^2) + bq^2, \text{ and } y = g(aq^2 + p^2) + 2afpq + bpq$$

And so we have solved the case where $ax^2 + bx + c = y^2$

2 Elements of Algebra: Part II, Chapter 7

2.1 Preliminary Steps

Conditions First Euler eliminates certain conditions for a in the equation $an^2 + 1 = m^2$:

- a cannot be a square (else we have a square plus one is a square which is never true for $m, n > 0$)
- a cannot be negative (else m would be in the complex plane)
- once we find one value n then we may find infinite values based on this value, thus we need only find the least such value

Historical Note It is worth noting here Lagrange attempts to make the conditions on coefficients etc. even more exact. A significant amount of time is devoted to this in the seventh chapter of his additions to Euler's book. In fact, the entirety of the first eight pages of this chapter is spent discussing different cases and constructions for coefficients of the Pell Equation.[2]

This book by Euler is quite famous for the fact that he made a rather significant error in attributing this equation to the mathematician John Pell:

> Pell, an English writer, has taught us to find [minimal solutions] by an ingenious method, which we shall here explain

But of course, Pell did not do this, in fact it was Lord Brouncker who gave the first general solution method in Europe. We will discuss this further in a later section.

2.2 General Method

Euler first offers almost a warning, that while there is a general method, each case must be done individually:

> This method is not such as may be employed generally, for any number a whatever; it is applicable only to each particular case.

That said, the general method is the same for each a, it just must be repeated an indeterminate amount of times before it yields the answer. Here we will give two examples given by Euler.

Example 2.1. Solving $5n^2+1 = m^2$ First Euler notes that $\sqrt{5n^2+1} > 2n$. Thus we write $\sqrt{5n^2+1} = 2n+p$ or $5n^2 + 1 = 4n^2 + 4np + p^2$. This yields $n^2 = 4np + p^2 - 1 \implies n = 2p + \sqrt{5p^2 - 1}$.

Now (as before) we know that $\sqrt{5p^2 - 1} > 2p \implies n > 4p \implies n = 4p + q$ for some q. This gives us $2p + q = \sqrt{5p^2 - 1}$, by squaring both sides we get: $4p^2 + 4pq + q^2 = 5p^2 - 1 \implies p^2 = 4pq + q^2 + 1$. Finally this yields $p = 2q + \sqrt{5q^2 + 1}$. Euler then says that as $q = 0$ will allow $p > 0$ we take $q = 0 \implies p = 1 \implies n = 4 \implies m = 9$. To verify we check: $5(4^2) + 1 = (9)^2 \implies 81 = 81$.

Example 2.2. Solving $7n^2 + 1 = m^2$[Method 1]

- Here Euler immediately notices that $m > 2n$ and thus makes $m = 2n + p$. With this we get

  $$7n^1 + 1 = 4n^2 + 4np + p^2, \text{ or } 3n^2 = 4np + p^2 + 1.$$

  This in turn yields $n = \dfrac{2p + \sqrt{7p^2 - 3}}{3}$. Since $n > \frac{4}{3}p > p$ let $n = p + q$ and thus $p + 3q = \sqrt{7p^2 - 3}$. Square both sides to get $p^2 + 6pq + 9q^2 = 7p^2 - 3 \implies 2p^2 = 2pq + 3q^2 + 1$.

  Thus $p = \dfrac{q + \sqrt{7q+2}}{2}$.

[2] See pages 550 to 558 in the John Hewlett Translation (1984).

- As before we notice that $p > \frac{3q}{2} > q$. So let $p = q + r$, we have $q + 2r = \sqrt{7q^2 + 2}$ which yields $q^2 + 4qr + 4r^2 = 7q^2 + 2 \implies 3q^2 = 2qr + 2r^2 - 1$.

  This gives us $q = \dfrac{r + \sqrt{7r^2 - 3}}{3}$.

- One last time we find that $q > r$, let us suppose that $q = r + s$, we get: $2r + 3s = \sqrt{7r^2 - 3}$. Square both sides to get $4r^2 + 12rs + 9s^2 = 7r - 3 \implies r^2 = 4rs + 3s^2 + 1$.

  Thus we are left with $r = 2s + \sqrt{7s^2 + 1}$.

- Now we notice that this formula is of the same form as the one we started with: $7n^2 + 1 = m^2$. This, according to Euler, is our cue to set $s = 0$ which implies $r = 1 \implies q = 1 \implies p = 2 \implies n = 3 \implies m = 8$. He then supplies the following alternate method which is much shorter and indicative of a more general solution strategy.

Example 2.3. Solving $7n^2 + 1 = m^2$[Method 2]

- Here we notice that $m < 3n$ (this is in contrast to the previous method whereby we noticed $m > 2n$). Thus we make $m = 3n - p$ which gives us $7n^2 + 1 = 9n^2 - 6np + p^2 \implies 2n^2 = 6np - p^2 + 1$.

  This yields $n = \dfrac{3p + \sqrt{7p^2 + 2}}{2}$.

- So $n > 3p$ thus (as before) we write $n = 3p - 2q$. This of course gives us $9p^2 - 12pq + 4q = 7p^2 + 2 \implies p^2 = 6pq - 2q^2 + 1$. This results in $p = 3q + \sqrt{7p^2 + 1}$

- Since as before this is of the same form as the original problem we set $q = 0$. This gives us $p = 1 \implies n = 3 impliesm = 8$ the same solution as in the first method but one cycle shorter. Thus by considering additive residues *and* subtractive residues one may significantly decrease the number of calculations in longer problems of this type.

Reflections on Euler's Method Euler gives one more example ($a = 8$ which is trivial) and then before he delves into the behemoth that is $a = 13$ (requiring no less than 11 cycles) he reflects upon his method. I will let Euler speak for himself:

> We may proceed, in the same manner, for every other number, a, provided it be positive and not a square; and we shall always be led, at least, to a radical quantity, such as $\sqrt{at^2 + 1}$, similar to the first, or given formula, and then we have only to suppose $t = 0$; for the irrationality will disappear, and by tracing back the steps, we shall necessarily find such a value of n, as will make $an^2 + 1$ a square.

After tackling the case of $a = 13$ he promises the reader a table of the first values for m and n for all $2 \le a \le 100$. He then goes on to prove the case where $a = b^2 \pm 1$ or $a = b^2 \pm 2$. At the end of the chapter we notice a note from Lagrange:

> Our author might have added here another very obvious case, which is when a is of the form $e^2 \pm \frac{2}{c}e$; for then by making $n = c$ our formula $an^2 + 1$ becomes $e^2c^2 \pm 2ce + 1 = (ec \pm 1)^2$. [. . .] And as a great many numbers are included in the above form, I have been induced to place it here, as a means of abridging the operations in these particular cases.
>
> The reader is indebted to MR. P. Barlow of the Royal Academy, Woolwich, for the above note; . . .

It is worth noting that while comment was made a mere six years after publication of the original (1771), so Euler would have been alive at this time, he was also almost completely blind, thus preventing him from reading, unaided, this comment along with all of Lagrange's additions. The work of Lagrange on this book is, in some sense, a testament to the closeness of their association.

3 Lagrange and His additions to Euler

Chapter Seven In this chapter of the Additions, Lagrange considers special cases of the coefficients for a and other types of quadratic Diophantine equations.[3] He begins with a discussion of all equations of the form $\sqrt{ax^2+1} = y \in \mathbb{Q}$, during which he goes over all of the cases and forms of a and y. One of the major concepts of this chapter is the idea that all quadratics of certain general forms can be reduced down to the form $ax^2 + b = y^2$ which can be further reduced to an instance of Pell's equation. He also proposes that there are equations of this form that are not solvable for integer numbers. Eg: $x^2 - 79y^2 = 101$. This is a contradiction of one of Euler's proofs: (note, spelling maintained)

> M. Euler, in an excellent Memoir printed in Vol.IX. of the *New Commentaries of Petersburg*, finds by induction this rule for determining the resolvibility of every equation of the form $x^2 - Ay^2 = B$, when B is a prime number. It is, that the last equation must be possible, whenever B shall have the form $4An + r^2$, or $4An + r^2 - A$; but the foregoing example shews this rule to be defective for 101 is a prime number, of the form $4An + r^2 - A$ making $A = 79, n = -4,$ and $r = 38$; yet the equation , $x^2 - 79y^2 = 101$, admits of no solution in whole numbers. [. . .]

Chapter Eight Introduction Lagrange spends the beginning of this chapter crediting Euler and other Mathematicians for their contributions toward the solving of the Pell Equation. It is worth noting that he does *not* incorrectly attribute the equation to Pell. Instead, he credits Fermat as the originator of this concept of reducibility of quadratic forms, but he gives Euler the following honour:

> However, we are indebted to Euler alone for the remark, that this problem [$p^2 = Aq^2 + 1$] is necessary for finding all the possible solutions of [indeterminate quadratic] equations.

Lagrange then mentions that while we may be able to find one solution to such an equation we may not be able to attain other solutions from the first (this was proven more formally in chapter Seven). He then goes on to say:

> With regard to the manner of resolving equations of the form $p^2 = Aq^2 + 1$, I think that of Chap. VII [see previous section], however ingenious it may be, is still far from perfect. For, in the first place, it does not show that every equation of this form is always resolvable in whole numbers, when a is a positive number not a square. Secondly, it is not demonstrated, that is must always lead to the solution sought for.

He then goes on to bash Wallis' proof and claims that his is far superior. Admittedly, his proof is only 5 pages long–and is quite elegant. We will include part of it below.

3.1 Continued Fraction Method

Lagrange (like the Indian mathematicians before him) proves in an earlier section that approximations of $\sqrt{A}$ can be written as continued fractions. He then shows that this is equivalent to finding $\frac{p_n}{q_n}$ where each of these is a better approximation of $\sqrt{A}$.

Continued Fraction Definition Lagrange takes a continued fraction to be of the following form and notation (exactly what Lagrange wrote):

$$\mu + \cfrac{1}{\mu' + \cfrac{1}{\mu'' + \cfrac{1}{\cdots \mu^{\varsigma}}}}$$

[3]It is worth noting here that Lagrange's notation and presentation is significantly different from Euler's. Overall, he appears more hurried, less organized. The proofs lack the clarity and concision of Euler's especially with respect to the cyclic nature of their respective methods. Part of this is subjective of course, but to me a different letter is preferable to the same letter with three primes any day.

where μ^ς is a term in the infinite series of $\mu', \mu'' \cdots$. Where all such μ are positive without loss of generality.

Use of Residues Since p and q are coprime we know that $\gcd(p, q) = 1$. We also know that each level of this continued fraction will have the form of some number times μ^ς plus a residue equals the numerator for some ς. What we mean is:

$$p = \mu q + r,\ q = \mu' r + s,\ r = \mu'' s + t\ etc.$$

If we go on like this the last remainder will be 0 and the previous remainder will be 1 (as the original numbers are coprime). "Thus, μ will be the approximate integer value of $\frac{p}{q}$, μ' that of $\frac{q}{r}$" etc. since each value is less than the true value (except for μ^ς).

For this, if we take μ^ς we let $\frac{p}{q} = \sqrt{A} \implies p^2 - Aq^2 = 0$. So first we substitute $\mu q + r = p$ so $(\mu^2 - A)q^2 + 2qr\mu + r^2 = 0$, now we approximate $\frac{q}{r}$ by

$$(\mu - A)\left(\frac{q}{r}\right)^2 + 2\mu\left(\frac{q}{r}\right) + 1 = 0$$

Now we use the fact that $\mu' r + s = q$, and the above equation becomes

$$(\mu^2 - A)q^2 + 2\mu qr + r^2$$

The root of this equation is $\frac{r}{s}$, thus we approximate to get the value of μ''. Again we let $\mu'' r + s = r$ etc. Since all of the μ^i are greater than zero and decreasing, we know that we will eventually hit the magic μ^ς case where the remainder is zero. Now let t be the last remainder (and thus equal to zero) and s be the next to last remainder (and thus equal to 1, as stated before). We only have to continue these operations inasmuch as the first term has a coefficient not equal to 1. At that step, we take $r = 1$ and $s = 0$ and then continue working backwards to the corresponding values of p and q, as done by Euler in the previous section.

Issues with Euler's Method This is essentially a reverse analysis of Euler's method. The implication is that it should not matter whether we take a remainder μ that is positive or negative so long as we decrease eventually to 0. On this matter Lagrange states:

> Wallis also expressly says, that we may employ the limits for $\mu, \mu', \mu'' \cdots$ either in *plus* or *minus*, at pleasure; and he even proposes this, as the proper means often of abridging calculation. This is likewise remarked by Euler, Art. 102, *et seq.* of the chapter just now quoted.
>
> However, the following example with shew that by setting about it in this way, we may run the risk of never arriving at the solution of the equation proposed.

Example 3.1. Solving $p^2 = 6q^2 + 1$ First we have $p = \sqrt{6q^2 + 1}$. Lagrange neglects the 1 and takes $p = q\sqrt{6}$. Now we take the fact that $2 < \sqrt{6} < 3$ and set $\frac{p}{q} = \sqrt{6}$. Take the limit "in *minus*", and let $\mu = 2$. This implies that $p = 2q + r \implies -2q^2 + 4qr + r^2 = 1$.

Method 1 This in turn implies $q = \dfrac{2r + \sqrt{6r^2 - 2}}{2}$. Lagrange again ignores the constant and takes $q = \dfrac{2r + r\sqrt{6}}{2}$. Thus we have the second term: $\dfrac{q}{r} = \dfrac{2 + \sqrt{6}}{2} \implies 2 < \dfrac{q}{r} < 3$ Now we take the negative limit again and let $q = 2r + s$. This implies $r^2 - 4rs - 2s^2 = 1$. Thus we may have $s = 0$ and $r = 1 \implies q = 2 \implies p = 5$

Method 2 Now let us go back to $-2q^2 + 4qr + r^2 = 1$. If we take the limit "in *plus*" instead of "in *minus*" as before we let $q = 3r - s$ for $s > 0$. Thus we have $-5r^2 + 8rs - 2s^2 = 1$.

This yields $r = \dfrac{4s + \sqrt{6s^2 - 5}}{5}$. Again, we neglect the constant and have $r = \dfrac{4s + \sqrt{6}}{5}$. As a result we have $\dfrac{r}{s} = \dfrac{4 - \sqrt{6}}{5} \implies 1 < \dfrac{r}{s} < 2$.

Take the positive limit again and we get $\frac{s}{t} = \frac{6+\sqrt{6}}{6} \implies 1 < \frac{s}{t} < 2$ Take it once more and we get $\frac{t}{u} = \frac{6+\sqrt{6}}{5} \implies 1 < \frac{t}{u} < 2.$

What We Learn From this Example We can continue doing this indefinitely and we know (though Lagrange does not say this) that the sequence will be cyclic and thus will never converge to such a value. He then says that the same must happen when we start with *plus* and take the rest *minus*. Thus he has shown a significant flaw in both Euler's and Wallis' method; one may not simply take the residues in what ever operations one wishes but must rather be consistent with one's direction. This is an especially interesting result as it was one that was known (tacitly) to Indian mathematicians six centuries prior: as the Indian mathematicians only considered positive residues, especially in the Chakravalah method, they would never have faced this troubling conclusion.

Conclusions

In Euler's *Elements of Algebra*, we see an attempt to coalesce the study of basic Analysis, Algebra, and Number theory into a structured whole. The fluidity of Euler's methods attests to the beautiful interconnectedness of these fields. When we put together Euler's *Elements of Algebra* with Lagrange's *Additions*, we have a piece of mathematical literature that captures the dialogue of mathematics as it was in the eighteenth century. [4]

[4] Attatched please find a series of examples and notes concerning the methods of Brahmagupta and Bhaskara II for solving the Pell equation and attaining further solutions.

References

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4 Notes On Indian Mathematicians Attempts at Solving the Pell Equation

4.1 Preliminary Steps

The person responsible for the earliest work regarding the generalized equation is Brahmagupta.

4.1.1 Solving $92x^2 + 1 = y^2$

> Put down twice the square root of a given square multiplied by a multiplier and increased or diminished by an arbitrary number

So we take:

$$92x^2 + b = y^2$$
$$92(1)^2 + 8 = (10)^2$$

so we have: $x_0 = 1,\ y_0 = 10,\ b_0 = 8$ Now he writes the solution in two rows:

$$\begin{array}{ccc} x_0 & y_0 & b_0 \\ x_0 & y_0 & b_0 \end{array}$$

now Brahmagupta says:

> The product of the fist pair, multiplied by the multiplier, with the product o the last pair, is the [new] last root.

which means we take the sum of the squares of roots multiplied by the constant multiplier and let that that be the new final root, or y_1: $Dx_0^2 + y_0^2 = y_1$

Brahmagupta then prescribes:

> And the The sum of the thunderbolt products [cross multiplication] is the [new] first root.

We also take the cross multiplication of the two first roots, or $x_0y_0 + x_0y_0 = 2x_0y_0 = x_1$.

> The additive is equal to the product of the additives.

so the new additive factor is $b_1 = b_0^2$.

This leads to the conclusion that

$$92x^2 + 64 = 192^2$$

has solutions $(20, 192)$ for (x_1, y_1) This is easy enough to verify but it points to a deeper result. Namely, that for any equation of the following form we have:

$$Dx_0^2 + b_0 = y_0^2 \implies D(2x_0y_0)^2 + (b_0)^2 = (Dx_0^2 + y_0^2)^2$$

From this Brahmagupta composed these solutions s_1, s_0 to get:

Theorem 4.1 (One of Brahmagupta's Contributions)**.**

$$\textit{Given } Du_0^2 + c_0 = v_0^2 \textit{ and } Du_1^2 + c_1 = v_1^2$$
$$D(u_0v_1)^2 + c_0c_1 = (Du_0u_1 + v_0v_1)^2$$

Brahmagupta concluded:

> The two square roots, divided by the [original] additive or subtractive, are the [roots for] additive unity.

So we have $(\frac{20}{8}, \frac{192}{8}) = (\frac{5}{2}, 24)$ as a solution for additive one. And we are left with:

$$92\left(\frac{5}{2}\right)^2 + 1 = (24)^2$$

4.1.2 The Kicker:

Now Brahmagupta composes this soluton with itself, so we have:

$$Dx_0^2 + b_0 = y_0^2 \implies D(2x_0y_0)^2 + (b_0)^2 = (Dx_0^2 + y_0^2)^2$$

from before and just plug and chug:

$$92(2\left(\frac{5}{2}\right)(24))^2 + (1)^2 = (92\left(\frac{5}{2}\right)^2 + (24)^2)^2$$

$$\implies 92(5 \cdot 24)^2 + (1)^2 = (23 \cdot 4 \cdot \frac{25}{4} + (24)^2)^2$$

$$\implies 92(120)^2 + 1 = (23 \cdot 25 + (24))^2 = (1151)^2$$

This composition is limited but when it finds a solution it enables one to find infinitely many solutions.

4.2 Solving solution for Additive 4

Brahmagupta also gave the solution for special cases of additive four a (and simillarly for subtractive four.

If $v$is odd or u is even then

$$(u_1, v_1) = \left(u\left(\frac{v^2-1}{2}\right), v\left(\frac{v^2-3}{2}\right)\right)$$

if v is even and u is odd:

$$(u_1, v_1) = \left(\frac{2uv}{4}, \frac{Du^2+v^2}{4} = \frac{2v^2-4}{4}\right)$$

4.3 Notes:

We must note that none of these facts are proven by Brahmagupta, nor do we know how this method was discovered or in relation to what. This tradition of Indian mathematics was studied extensively by many indian mathematicians and was solved completely by Acarya Jayadeva (c. 1000) almost eight centuries before Lagrange proved the cyclic method terminates for all values.

4.4 Brahmagupta's Identity and its Development into the Chakravala Method

Brahmagupta's method which we saw led to the following identity

$$(x_1^2 + ny_1^2)(x_2^2 - ny_2^2) = (x_1x_2 \pm ny_1y_2)^2 + n(x_1y_2 \mp y_1x_2)^2$$

meant that given any one solution to the Pell's equation we could find infinitely many others. Thus it is important to find the minimal solution.

Following Brahmagupta we have Bhaskara's rule for the general Pell's Equation.

4.4.1 Bhaskara's rule and solving $67x^2 + 1 = y^2$

Bhaskara states:

> Making the smaller and a larger roots and the additive into the dividend, the additive, and the divisor, the multiplier is to be imagined.

So we choose a pair for any additive b as we did for Brahmagupta's method. Let us here take $(x_0, y_0) = 8$ and $b_0 = -3$ Now solve th indeterminate equations $um + v = bn$ for m, or $m + 8 = -3n$. The result is $m = 1 + 3t, n = -3 - t$ for $t \in \mathbb{Z}$

When the square of the multiplier $[m^2]$ is subtracted from the "nature" $[D$ or is diminished by the "nature" so that the remainder is small, that divided by the additive $[b]$ is the new additive.

So we must take t so that m^2 is as close to D as possible, and

$$b_1 = \pm\frac{D - m^2}{b} = \pm\frac{D - (1 + 6t + 9t^2)}{b}$$

Note that this b_1 may be negative.

The quotient of the multiplier is the smaller square root; from that is found the greatest root.

Taking the indeterminate form $um + v = bn$ as seen before we solve for n to get the first root $u_1 = \frac{um+v}{b}$ and substitute u_1 to get the second: $\sqrt{Du_1^2 + b_1}$. Thus we have:

$$(u_0, v_0, b_0) \implies \left(\frac{u_0 m + v_0}{b_0}, \sqrt{D\left(\frac{u_0 m + v_0}{b_0}\right)^2 \pm \frac{D - m^2}{b}}, \pm\frac{D - m^2}{b}\right)$$

In our example we get: $t = 2 \implies m = 7 \implies m^2 = 49$ which is close to $D = 67$ Thus we have

$$\left(\frac{u_0 m + v_0}{b_0}, \sqrt{D\left(\frac{u_0 m + v_0}{b_0}\right)^2 \pm \frac{D - m^2}{b_0}}, \pm\frac{D - m^2}{b}\right)$$

$$\implies \left(\frac{1 \cdot 7 + 8}{-3}, \sqrt{67\left(\frac{1 \cdot 7 + 8}{-3}\right)^2 - \frac{67 - 7^2}{-3}}, -\frac{67 - 7^2}{-3}\right) = \left(\frac{15}{-3}, \sqrt{67 \cdot 5^2 + 6}, \frac{18}{3}\right) = (5, 41, 6)$$

Then it is done repeatedly, leaving aside the previous square roots and additives. They call this the *chakravla* (cycle). Thus there are two integer square roots increased by four, two or one. The supposition for the sake of an additive one is from the roots with four and two additives.

Thus Bhaskara states that if this chakravala process is iterated then we will eventually get an equation with $b_n = \pm 4, \pm 2, or \pm 1$. We know from our previous forays with Brahmagupta that equations with these additives are solvable by the method shown in the first three subsections. First we will finish this example then we will move on to deeper questions.

So to continue with our example of $67x^2 + 1 = y^2$.

We began with $67 \cdot 1 - 3 = 8^2$ and then progressed on to $67 \cdot 5^2 + 6 = 41^2$ and thus $(u_1, v_1, b_1) = (5, 41, 6)$. Now we must solve $5m + 41 = 6n$ with $|m^2 - 67|$. So we take $m = 5$ and get

$$\left(\frac{u_1 m + v_1}{b_1}, \sqrt{D\left(\frac{u_1 m + v_1}{b_1}\right)^2 \pm \frac{D - m^2}{b_1}}, \pm\frac{D - m^2}{b}\right)$$

$$\implies \left(\frac{5 \cdot 5 + 41}{6}, \sqrt{67\left(\frac{5 \cdot 5 + 41}{6}\right)^2 \pm \frac{67 - 5^2}{6}}, \pm\frac{67 - 5^2}{6}\right)$$

$$= \left(\frac{66}{6}, \sqrt{67\left(\frac{66}{6}\right)^2 \pm \frac{42}{6}}, \pm\frac{42}{6}\right) = (11, \sqrt{67 \cdot 11^2 - 7}, -7) = (11, 90, -7)$$