189-346/377B: Number Theory

Assignment 5

Due: Monday, March 21

1. An integer n is said to be *square-free* if its prime factorisation is of the form

$$n = p_1 p_2 \cdots p_r,$$

where $p_1, \ldots, p_r$ are *distinct* primes. Show that for all real $s > 1$,

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n \in S} \frac{1}{n^s},$$

where

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

is the Riemann zeta function, and S is the set of positive square free integers.

2. Using a Sieve argument (or otherwise), show that the number of square-free integers that are less than or equal to x is equal to

$$\zeta(2)^{-1} x + o(x).$$

3. Show that any integer of the form $4n+3$ always has a prime divisor of the form $4k+3$. Use this to give a proof that there are infinitely many primes of the form $4k+3$, analogous to Euclid's proof of the infinitude of primes that was recalled in class. Show by a similar argument that there are infinitely many primes of the form $3k+2$.

4. Let d be a prime. Show that any prime p which does not divide d but divides the integer

$$n^{d-1} + n^{d-2} + \cdots + 1$$

($n \in \mathbf{Z}$) is necessarily of the form $4d+1$. Use this to show that there are infinitely many primes of the form $4d+1$. (Hint: assume otherwise, and study the asymptotics of $\#\{n^{d-1}+\cdots+n+1, \quad n \leq x^{1/d}\}$ as $x \longrightarrow \infty$ in two different ways to derive a contradiction.)

The following exercises are taken from the textbook by Levesque.

5. (Section 6.2, exercise 7 from Levesque.)
Show that, for all $s > 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1,$$

where $\mu(n)$ is the Möbius function defined by $\mu(n) = (-1)^t$ if n is a product of t distinct primes, and $\mu(n) = 0$ if t is divisible by the square of some prime.

6. Show that if $f(x)$ is a continuous, monotonically decreasing function which tends to 0 as $x \longrightarrow \infty$, and if the series $\sum_{n=1}^{\infty} f(n)$ diverges, then the function

$$F(n) := \sum_{j=1}^{n} f(j)$$

satisfies

$$F(n) \sim \int_1^n f(x)dx.$$

7. (Section 6.4, exercise 9 from Levesque.)
Let $\log_k x$ be the k-th iterate of the logarithm function, defined recursively by

$$\log_1 x = \log x, \qquad \log_k x = \log \log_{k-1} x.$$

Is there a continuous increasing function $f(x)$ such that $\lim_{x\to\infty} f(x) = \infty$, yet $f(x) = o(\log_k x)$ for all $k \geq 1$? If so, exhibit such a function.

Math 377 only:
8. Section 6.8., exercise 4 in Levesque.