189-346/377B: Number Theory
Assignment 1

Due: Monday, January 15

1. Let $M_n = 2^n - 1$ denote the nth Mersenne number. Show that if r divides n, then M_r divides M_n.

2. Show that if $2^k + 1$ is prime, then k must be a power of two. Write down $2^{2^n} + 1$ for $n = 0, 1, 2, 3$, and 4, and show that these numbers are prime.

377: Show that, if p is a prime divisor of $2^{2^n} + 1$, then p is of the form $p = 2^{n+1}k + 1$.

3. Let $li(x) = \int_2^x \frac{dt}{\log(t)}$ be the function that occurs in Gauss's statement of the Prime Number Theorem. Show that

$$\lim_{x \longrightarrow \infty} \frac{li(x)}{x/\log(x)} = 1,$$

and conclude that $li(x)$ can be replaced by the simpler function $x/log(x)$ in the statement of the PNT.

4. Show that 55 can be written as a difference of two perfect squares in exactly two different ways, and write down those expressions.

The following two questions will require you to use a computer algebra system.

5. Show that $2^{2^k} + 1$ is not prime for $k = 5, 6, 7$, and 8, by producing the factorisation. This is a calculation that Fermat was not able to do, since he did not have access to a computer! Based on the outcome of question 2, he rashly predicted that $2^{2^n} + 1$ is prime for all n; it is now believed that in fact this number is *never* prime for $n \geq 5$, a statement that seems a likely candidate for a true fact that could forever elude proof. For $n \geq 9$, it becomes

more dificult to factor these numbers. The fact that $2^{2^9}+1$ is not prime was known for quite some time, but its full factorisation is a computational *tour de force* that was only achieved in 1990. The full factorisation of $2^{2^{12}}+1$ is still not known; this number has a composite factor with 1187 digits which has not been factored to this day.

6. Compute $e^{\pi\sqrt{163}}$ with 30 significant digits on Pari. (For this, enter the Pari command $\backslash p$ 30.) What do you observe? Repeat the calculation with 40 significant digits. (This exercise is meant to get you familiar with using Pari, and also as a cautionary tale about drawing conclusions too hastily based on experimental data.)