A Reformulation of Dirichlet's Primes in Arithmetic Progression

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Abstract

This paper presents a reformulation of Dirichlet's theorem on Primes in Arithmetic Progressions, which states that for a and b relatively prime, the sequence $\{a+bx\}_{x=1}^{\infty}$ contains infinitely many primes. It is possible to reduce this well-known problem to a simply stated conjecture. Once assumed, this conjecture allows for completion of the theorem while avoiding most complex analysis. This much more elementary method would then only involve dual groups and the complex character function, simple series analysis and some linear algebra. I will state the conjecture and show how the proof follows elegantly from it.

1 Introduction

Let $G_b = (\mathbf{Z}/b\mathbf{Z})^*$ so that G_b refers to the invertible elements mod(b). Only integers of the form $a+bx$ where $(a,b)=1$ have a representative congruency in G_b. Therefore, it is clear that the cardinality of G_b is $\phi(b)$ (where $\phi(n)$ is the Euler phi function).

A *character* on G_b is an element in the *dual group* of G_b. The dual group of G_b is defined as

$$G_b^{\vee} \;=\; \{\; \textit{all homomorphisms } \chi : G_b \to \mathbf{C} - \{0\}\}$$

This yields multiplicative functions s.t. $\chi(n \cdot m) = \chi(n) \cdot \chi(m)$. Also notice that this group includes the trivial character χ_0 where $\chi(n) = 1 \; \forall n \in G_b$

A *Dirichlet character* modulo b is a modified character function from the positive integers to $\mathbf{C}$ such that, for all elements $a \in \mathbf{Z}$ s.t. $(a,b) > 1, \chi(a) = 0$. Other properties of dual groups will be discussed further on.

The next tool used is the powerful Euler Product. It is easy to prove the following equality:

$$\prod_{\text{primes } p<x} \frac{1}{1-\frac{1}{p}} \;=\; \sum_{n \;x\text{-smooth}} \frac{1}{n} \qquad \text{for some } x \geq 3 \;,\; x \in \mathbf{R}$$

Due to this equality and the multiplicativity of the Dirichlet character function, we get:

$$(*) \quad \prod_{p<x} \frac{1}{1-\frac{\chi(p)}{p}} \;=\; \sum_{n\ x\text{-smooth}} \frac{\chi(n)}{n} \quad \text{for all } \chi \in G_b^\vee$$

This sum is true for all finite values of x, but since we desire to show that the left-hand side converges for all p (with any non-trivial χ), it becomes necessary to take some sort of limit. Since we don't yet know if the right-hand side converges for $x \to \infty$ or even if a limit exists, we can consider the following modification:

$$\sum_{n \le x} \frac{\chi(n)}{n}$$

This sum can be shown to have a limit and converge *conditionally.* Note that we cannot take any rearrangement of this sum and presume a preservation of convergence.

In an attempt to compare this modified summation with the product in equation $(*)$, we proceed to split the summation as so:

$$\prod_{p<x} \frac{1}{1-\frac{\chi(p)}{p}} \;-\; \sum_{n \le x} \frac{\chi(n)}{n} \;=\; \sum_{n\ x\text{-smooth},\ n>x} \frac{\chi(n)}{n}$$

As long as the series and product are bounded using $x < +\infty$, this equality is ensured. A method of achieving the desired result requires the following conjecture on the summation of the Dirichlet L-function for $s = 1$ over the x-smooth numbers greater than x (for all characters other than the trivial):

$$\sum_{n\ x\text{-smooth},\ n>x} \frac{\chi(n)}{n} \;\to\; 0 \quad \text{as } x \to \infty$$

It is now plain to see that if the conjecture is true, and this difference gets very small as x gets large, then we will obtain:

$$\prod_{p} \frac{1}{1-\frac{\chi(p)}{p}} \;=\; \sum_{n=1}^{\infty} \frac{\chi(n)}{n}$$

This result will be shown to complete the proof.

2 The Dual of G_b

Here we will show that the character functions form an orthonormal basis for the space of functions from G_b to $\mathbf{C}$.

Definition 2.1: *$G_b^\vee$ is the set of all homomorphisms from the group G_b to the complex plane under multiplication.*

These functions are intuitively easy to construct but difficult to generate while working with non-cyclic groups.

Definition 2.2: *Let χ be a homomorphism from $G_b \to \mathbf{C}^*$. The corresponding Dirichlet character mod(b) is defined as the function $\chi' : \mathbf{Z} \to \mathbf{C}$ with the properties that: for all elements a in $\mathbf{Z}$ such that $(a,b) > 1,\ \chi'(a) = 0$ and for all elements c in $\mathbf{Z}$ such that $(c,b) = 1,\ \chi'(c) = \chi(c\ mod(b))$.*

These functions behave the same as the characters in the dual of G_b. Specifically, it is easy to show that a Dirichlet character is also multiplicative.

Example 2.1: The two Mod(4) Dirichlet characters are the trivial χ_0 and:

$$\chi(n) = \begin{cases} 0 & n \equiv 0, 2 \bmod(4) \\ 1 & n \equiv 1 \\ -1 & n \equiv 3 \end{cases}$$

Example 2.2: The six Mod(7) Dirichlet characters are:

$$\chi_1(n) = \begin{cases} 0 & n \equiv 0 \bmod(7) \\ 1 & n \equiv 1 \\ e^{\frac{2i\pi}{3}} & n \equiv 2 \\ e^{\frac{i\pi}{3}} & n \equiv 3 \\ e^{\frac{4i\pi}{3}} & n \equiv 4 \\ e^{\frac{5i\pi}{3}} & n \equiv 5 \\ -1 & n \equiv 6 \end{cases} \qquad \chi_2(n) = \begin{cases} 0 & n \equiv 0 \bmod(7) \\ 1 & n \equiv 1 \\ e^{\frac{4i\pi}{3}} & n \equiv 2 \\ e^{\frac{2i\pi}{3}} & n \equiv 3 \\ e^{\frac{2i\pi}{3}} & n \equiv 4 \\ e^{\frac{4i\pi}{3}} & n \equiv 5 \\ -1 & n \equiv 6 \end{cases}$$

$$\chi_3(n) = \begin{cases} 0 & n \equiv 0 \bmod(7) \\ 1 & n \equiv 1 \\ 1 & n \equiv 2 \\ -1 & n \equiv 3 \\ 1 & n \equiv 4 \\ -1 & n \equiv 5 \\ -1 & n \equiv 6 \end{cases} \qquad \chi_4(n) = \begin{cases} 0 & n \equiv 0 \bmod(7) \\ 1 & n \equiv 1 \\ e^{\frac{2i\pi}{3}} & n \equiv 2 \\ e^{\frac{4i\pi}{3}} & n \equiv 3 \\ e^{\frac{4i\pi}{3}} & n \equiv 4 \\ e^{\frac{2i\pi}{3}} & n \equiv 5 \\ -1 & n \equiv 6 \end{cases}$$

$$\chi_5(n) = \begin{cases} 0 & n \equiv 0 \bmod(7) \\ 1 & n \equiv 1 \\ e^{\frac{4i\pi}{3}} & n \equiv 2 \\ e^{\frac{5i\pi}{3}} & n \equiv 3 \\ e^{\frac{2i\pi}{3}} & n \equiv 4 \\ e^{\frac{i\pi}{3}} & n \equiv 5 \\ -1 & n \equiv 6 \end{cases} \qquad \chi_0(n) = \begin{cases} 0 & n \equiv 0 \bmod(7) \\ 1 & n \equiv 1 \\ 1 & n \equiv 2 \\ 1 & n \equiv 3 \\ 1 & n \equiv 4 \\ 1 & n \equiv 5 \\ 1 & n \equiv 6 \end{cases}$$

Note that there are as many character functions as there are elements in the group $G_b = (\mathbf{Z}/b\mathbf{Z})^*$ (I will prove that this is true in general). Also note that $G_7^\vee$ is cyclic, as is G_7 itself. (As an explanation, please refer to Ireland and Rosen [3] for details on the isomorphism between a group and its dual.) Thirdly, what is referred to in this last example as χ_3 is more commonly known as the

Legendre symbol modulo 7.

Proposition 2.1: $G_b^\vee$ *is a group under multiplication.*
Proof:
Multiplication of characters is associative, and χ_0 is the identity. To show closure and the existence of inverses, if $\chi_1, \chi_2 \in G_b^\vee$, define $\chi_1\chi_2$ as the function that takes $a \in G_b$ to $\chi_1(a) \cdot \chi_2(a)$. Then, we can define the inverse $\forall\ \chi \in G_b^\vee$ as $\chi^{-1}(a) := [\chi(a)]^{-1} \in \mathbf{C}^*$. This χ^{-1} is a homomorphism from the group to $\mathbf{C}^*$ and is therefore also in $G_b^\vee$. Closure is also clear from the fact that the characters are all homomorphisms.

Lemma 2.1: *For* $\chi(n) \neq \chi_0(n)$, $\sum_{n\in G_b} \chi(n) = 0$. *And* $\sum_{n\in G_b} \chi_0(n) = \phi(b)$.
Proof: Choose $a \ \epsilon \ G_b$ s.t. $\chi(a) \neq 1$

$\chi(a) \sum_{n\epsilon G_b} \chi(n) = \sum_{n\epsilon G_b} \ \chi(an) = \sum_{n\epsilon G_b} \chi(n)$

$\Rightarrow \chi(a) \sum_{n\epsilon G_b} \chi(n) = \sum_{n\epsilon G_b} \ \chi(n)$

$\Rightarrow 0 = \sum_{n\epsilon G_b} \chi(n)(1 - \chi(a))$

$\Rightarrow 0 = \sum_{n\epsilon G_b} \chi(n)$

The second part is obvious as $\mathrm{card}(G_b) = \phi(b)$.

Lemma 2.2: *The function from Func*$(G_b, \mathbf{C})^2 \to \mathbf{C}$ *defined by*

$$< \chi, \psi > = \frac{1}{\phi(b)} \sum_{n\in G_b} \chi(n)\overline{\psi(n)} \quad \textit{where } \phi(n) \textit{ is the Euler phi function}$$

is a Hermitian inner product on this vector space.
Proof:
Clearly, $<\chi, \phi>=\overline{<\phi, \chi>}$ and for $\lambda \in \mathbf{C}$, $\lambda<\chi, \psi>=<\lambda\chi, \psi>=<\chi, \overline{\lambda}\psi>$.
All that remains to show is positivity, which follows from:

$$< \chi, \chi >= \sum_{n\in G_b} \frac{\chi(n)\overline{\chi(n)}}{\phi(b)} = \sum_{n\in G_b} \frac{\Re[\chi(n)]^2 + \Im[\chi(n)]^2}{\phi(b)} > 0$$

and is zero $\Leftrightarrow \Re[\chi(n)] = 0$ and $\Im[\chi(n)] = 0$.

Lemma 2.3: *The* χ_i *are orthonormal in Func*$(G_b, \mathbf{C})$ *relative to this inner product.*
Proof: Let χ_i and χ_j be elements of $G_b^\vee$. Then

$$< \chi_i, \chi_j >= \sum_{n\in G_b} \frac{\chi_i(n)\overline{\chi_j(n)}}{\phi(b)} = \sum_{n\in G_b} \frac{\chi_i(n)\ \chi_j^{-1}(n)}{\phi(b)}.$$

If $\chi_i \neq \chi_j$ then $\chi_i(n) \cdot \chi_j(n)^{-1} \neq 1$ for some n, since inverses are unique in a group. And since $\chi_i, \chi_j \epsilon G_b^\vee \Rightarrow \chi_i \cdot \chi_j^{-1} \in G_b^\vee$ and, by Lemma 2.1

$$\sum_{n \in G_b} \frac{\chi_i(n)\chi_j(n)^{-1}}{\phi(b)} = 0$$

If $\chi_i = \chi_j$ then

$$\sum_{n \in G_b} \frac{\chi_i(n)\chi_j(n)^{-1}}{\phi(b)} = \sum_{n \in G_b} \frac{\chi_0(n)}{\phi(b)} = \sum_{n \in G_b} \frac{1}{\phi(b)} = \frac{\phi(b)}{\phi(b)} = 1$$

Lemma 2.4: *There exists a natural injection between G_b and its* bidual $G_b^{\vee\vee}$.
Proof: $G_b^{\vee\vee}$ is defined as the dual of the dual group, or as $G_b^{\vee\vee} = \mathrm{Hom}(G_b^\vee, \mathbf{C})$. We can see this bijection by defining the characters (ψ) of the bidual with respect to the elements of G_b:

$$\forall\ a \in G_b \quad a \rightarrow \psi_a(\chi) := \chi(a)$$

ϕ is a homomorphism the sends members of the dual group to $\mathbf{C}^*$. Supposing ϕ is not injective would imply that, for some a and a' in G_b,

$$\chi(a) = \chi(a') \text{ for all } \chi$$

$$\chi(a \cdot (a')^{-1}) = 1 \text{ for all } \chi$$

which implies that $a \cdot a'^{-1} = 1$ and therefore $a = a'$.

Theorem 2.1: *The elements of the dual group $G_b^\vee$ form an orthonormal basis for the functions from $G_b \rightarrow \mathbf{C}$.*
Proof: From Lemma 2.3 we get that the χ_i are lineraly independent and hence, $\sharp G_b^\vee \leq \sharp G_b$. Additionally, this gives us that $\sharp G_b^{\vee\vee} \leq \sharp G_b^\vee$ (since $G_b^{\vee\vee}$ is similarly the dual of $G_b^\vee$ and so the same inequality applies). From Lemma 2.4, we have that $\sharp G_b \leq \sharp G_b^{\vee\vee}$. Therefore, $\sharp G_b = \sharp G_b^\vee$.

Since $\sharp G_b$ =dim(Func($G_b, \mathbf{C}$)), the elements of the dual group form an orthonormal set with the same cardinality as the dimension of the space. Therefore, the χ_i's form an orthonormal basis over the space of functions.

Corollary 2.1: *There exists a bijection between G_b and $G_b^{\vee\vee}$.*

3 Euler's Equation and the *L*-Function at s=1

The desirable simplicity of this method comes from side-stepping the use of the complex s-variable in the L-function. After proving that $L(1, \chi)$ converges conditionally for all characters $\chi \neq \chi_0$, our conjecture will allow for completion of the theorem.

Proposition 3.1: *The Euler Equality*

$$\prod_{p<x}(\frac{1}{1-\frac{1}{p}}) = \sum_{n\ x\text{-smooth}} \frac{1}{n} \quad \text{where } p \text{ are prime}$$

Proof: The expression in the product represents the sum of a convergent geometric series, which gives us

$$\prod_{p<x}(1+\frac{1}{p}+\frac{1}{p^2}+...)$$

And where p_x is the largest prime less than x,

$$=(1+\frac{1}{2}+\frac{1}{4}+...)\cdot(1+\frac{1}{3}+\frac{1}{9}+...)\cdots(1+\frac{1}{p_x}+(\frac{1}{p_x})^2+...)$$

Finally, by the usual rules for multiplying sums, each resulting term will be a unique product of one 'choice' per bracket. By **N** being a UFD, each product will therefore yield a unique term.

$$=\sum_{n\ x\text{-smooth}} \frac{1}{n}$$

Proposition 3.2: *For any multiplicative function $f(x)$, the following equality holds:*

$$\prod_{p<x}(\frac{1}{1-\frac{f(p)}{p}}) = \sum_{n\ x\text{-smooth}} \frac{f(n)}{n}$$

Proof: This follows directly from the fact that $f(p^k) = f(p)^k$ as well as $f(p\cdot q) = f(p)\cdot f(q)$. Then it becomes clear that $\forall n,\ f(n) = f(p_1^{i_1}p_2^{i_2}...p_x^{i_x}) = f(p_1)^{i_1}f(p_2)^{i_2}...f(p_x)^{i_x}$

Theorem 3.1: *The harmonic series of prime numbers, $\sum_{p\ prime}\frac{1}{p}$, diverges.*
Proof: First, note that $\frac{1}{p}$ is asymptotic to $-\log(1-\frac{1}{p})$ for all p, meaning that

$$\lim_{p\to\infty}\frac{\frac{1}{p}}{\log(\frac{1}{1-\frac{1}{p}})} = 1.$$

This can be written as follows:

$$\log(\frac{1}{1-\frac{1}{p}}) \sim \frac{1}{p}$$

A stronger property is that the difference between the sums

$$|\sum_{p<x}\log(\frac{1}{1-\frac{1}{p}}) - \sum_{p<x}\frac{1}{p}|$$

is controllable. We get that

$$
\begin{aligned}
|\sum_{p<x} \log(\frac{1}{1-\frac{1}{p}}) - \sum_{p<x} \frac{1}{p}| \quad &\leq \quad |\sum_{p<x}\sum_{n\geq 2}\frac{1}{np^n}| \\
&\leq \sum_{p<x}\sum_{n\geq 2}|\frac{1}{p^n}| \\
&= \sum_{p<x}\frac{1}{p^2}\frac{1}{1-\frac{1}{p}} \\
&\leq \sum_{p\leq x}\frac{2}{p^2} \\
&\leq \sum_{n}\frac{2}{n^2} \quad \text{which converges by p-series.}
\end{aligned}
$$

And with Prop 3.1,

$$
\log\sum_{n=1}^{x} \frac{1}{n} < \log \sum_{n\ x\text{-smooth}} \frac{1}{n} = \log(\prod_{p<x} (\frac{1}{1-\frac{1}{p}})) = \sum_{p<x} \log(\frac{1}{1-\frac{1}{p}}) \sim \sum_{p<x} \frac{1}{p}
$$

(The sums and product are clearly not equal or converging to zero, so we can take consider their logarithm.) If $\sum \frac{1}{p}$ converged, so would the $\log(\prod_{p<x}(\frac{1}{1-\frac{1}{p}}))$ (by the limit comparison test from basic calculus). If this is so, then the $\log(\sum \frac{1}{n})$ over the x-smooth numbers would be bounded. However, this is a contradiction, as this series diverges as $x \to \infty$.

Theorem 3.2: *(Dirichlet's Test for convergence)*
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two infinite series. If $\{b_n\}$ converges monotonically to zero, and the partial sums of $\sum_{n=1}^{\infty} a_n$ are bounded, then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.
Proof: Please refer to Gordon [2], p.232.

Lemma 3.2: *The partial sums of $\sum_{n=1}^{\infty}\ \chi(n)$ (with the Dirichlet character χ) are bounded for $\chi \neq \chi_0$.*
Proof: Let $S_N = \sum_{n=1}^{N}\ \chi(n) = \chi(1) + \chi(2) + ... + \chi(N) = 0$
Some of these $\chi(a)$ are zero and the rest are ≤ 1 in absolute value. There are precisely $\phi(b)$ non-zero χ's, and so a very rough upper bound on the infinite series is:

$$
|\sum_{n=1}^{\infty}\ \chi(n)| \leq \phi(b)
$$

Proposition 3.3: *L(1,χ)=$\sum \frac{\chi(n)}{n}$ is conditionally convergent.*
Proof: We've already seen that the partial sums of $\sum\ \chi(n)$ are bounded for

non-trivial characters. Additionally, $\{\frac{1}{n}\} \to 0$ monotonically. By Theorem 3.2, the series converges. However,

$$\sum_{n} |\frac{\chi(n)}{n}| + \sum_{p|n} \frac{1}{p} > \sum \frac{1}{p} = +\infty$$

Since $\sum_{p|n} \frac{1}{p}$ is finite, $L(1,\chi)$ diverges in absolute value and is therefore only conditionally convergent.

Conjecture: *The sum* $\sum \frac{\chi(n)}{n}$ *over the* x*-smooth numbers greater than* x *converges to zero as* x *gets large.*

Proposition 3.4: *Assume the conjecture. Then*

$$\prod_{p<x} \frac{1}{1-\frac{\chi(p)}{p}} \to \lim_{x\to\infty} \sum_{n<x} \frac{\chi(n)}{n}.$$

Proof: From Prop 3.2,

$$\prod_{p<x} \frac{1}{1-\frac{\chi(p)}{p}} = \sum_{n\ x\text{-smooth}} \frac{\chi(n)}{n}$$

$$\prod_{p<x} \frac{1}{1-\frac{\chi(p)}{p}} = \sum_{n\le x} \frac{\chi(n)}{n} + \sum_{n\ x\text{-smooth},\ n>x} \frac{\chi(n)}{n}$$

$$\prod_{p<x} \frac{1}{1-\frac{\chi(p)}{p}} - \sum_{n\le x} \frac{\chi(n)}{n} = \sum_{n\ x\text{-smooth},\ n>x} \frac{\chi(n)}{n}$$

Then, by the conjecture,

$$\prod_{p<x} \frac{1}{1-\frac{\chi(p)}{p}} - \sum_{n\le x} \frac{\chi(n)}{n} \to 0 \quad as\ x \to \infty$$

$$\prod_{p\ \text{prime}} \frac{1}{1-\frac{\chi(p)}{p}} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n}$$

Proposition 3.5: $\sum \frac{\chi(p)}{p}$ *is conditionally convergent* $\forall \chi \neq \chi_0$.

Proof: Now we have all we need, since $\prod_p (1-\frac{\chi(p)}{p})^{-1} \neq 0$ and converges. It is still true that

$$|\sum_{p<x} \log(\frac{1}{1-\frac{\chi(p)}{p}}) - \sum_{p<x} \frac{\chi(p)}{p}| \le \sum_{p<x}\sum_{n\ge 2} |\frac{1}{np^n}| \le +\infty$$

as shown in the proof of Theorem 3.1. Therefore, using an extended logarithm (functioning over complex values), we get that

$$\log(\prod_{p} (\frac{1}{1-\frac{\chi(p)}{p}})) = \sum_{p} \log(\frac{1}{1-\frac{\chi(p)}{p}}) \sim \sum_{p} \frac{\chi(p)}{p} \quad \text{converges by LCT}$$

Theorem 3.3: *(Dirichlet's Theorem on Primes in Arithmetic Progressions) For any natural numbers a and b such that $(a,b)=1$, the sequence $\{a+bx\}_{x=1}^{\infty}$ contains infinitely many primes.*
Proof: Construct a function ψ such that, for some $a \in (\mathbf{Z}/b\mathbf{Z})^*$:

$$\psi(n) = \begin{cases} 1 & n \equiv a \\ 0 & \text{otherwise} \end{cases}$$

$\psi(n)$ is a function in the space $\text{Func}(G_b, \mathbf{C})$, so it is spanned by the χ_i's. So, for some c_i's (using $t := \phi(b)$), $\psi(n) = c_1\chi_1(n) + ... + c_{t-1}\chi_{t-1}(n) + c_0\chi_0(n)$. Since the characters form an orthonormal basis, we can determine the values of the coefficients using the inner product described in Lemma 2.2. We only care to observe that

$$c_0 =< \psi(n), \chi_0(n) >= \frac{1}{\phi(b)} = \frac{1}{t} \neq 0$$

Now, divide the linear combination by a prime p, and sum for every p. The expression becomes:

$$\sum_p \frac{\psi(p)}{p} = c_1 \sum_p \frac{\chi_1(p)}{p} + ... + c_{t-1} \sum_p \frac{\chi_{t-1}(p)}{p} + \frac{1}{t} \sum_p \frac{1}{p}$$

Each of the first $t-1$ sums converge (by Proposition 3.5) and the last sum diverges (by Theorem 3.1). And since

$$\sum_p \frac{\psi(p)}{p} = \sum_{p \equiv a} \frac{1}{p}$$

we have that $\sum_{p\equiv a} \frac{1}{p}$ diverges. Therefore, there are infinitely many primes congruent to a mod(b).

4 Numerical Analysis

Before advancing the conjecture, I did extensive numerical analysis (with PARI) on the sum, and modified the statement based on the results. For example, at first the statement involved two variables x_1 and x_2 with $x_2 \leq x_1$ as follows:

$$d(x_1, x_2) = \prod_{p<x_1} \frac{1}{1 - \frac{\chi(p)}{p}} - \sum_{n \geq x_2} \frac{\chi(n)}{n} = \sum_{n<x_2,\ n\ x_1\ smooth} \frac{\chi(n)}{n}$$

But when holding x_1 constant and modifying the smaller x_2, $d(x_1, x_2)$ is smallest when $x_1 = x_2$. (Please see figure 1.)

The fact that $d(x_1, x_2)$ only makes sense with $x_2 \leq x_1$ removed the possibility of a simple conclusion (if x_1 and x_2 were free or if $x_1 \leq x_2$, then we could easily have said that, since $\sum_{n\ x_1\text{-smooth}} \frac{\chi(n)}{n}$ converges, then the tail becomes arbitrarily small).

I also experimented with modifying the χ function (even within a mod group) but this choice has little importance as x gets large. In the appendix, I give an example summation using the mod 7 generator χ. It is clear that the sum tends drastically to zero amid diminishing tremors. (Please see figure 2.)

References

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3. K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, second ed., Springer-Verlag, New York, 1990.
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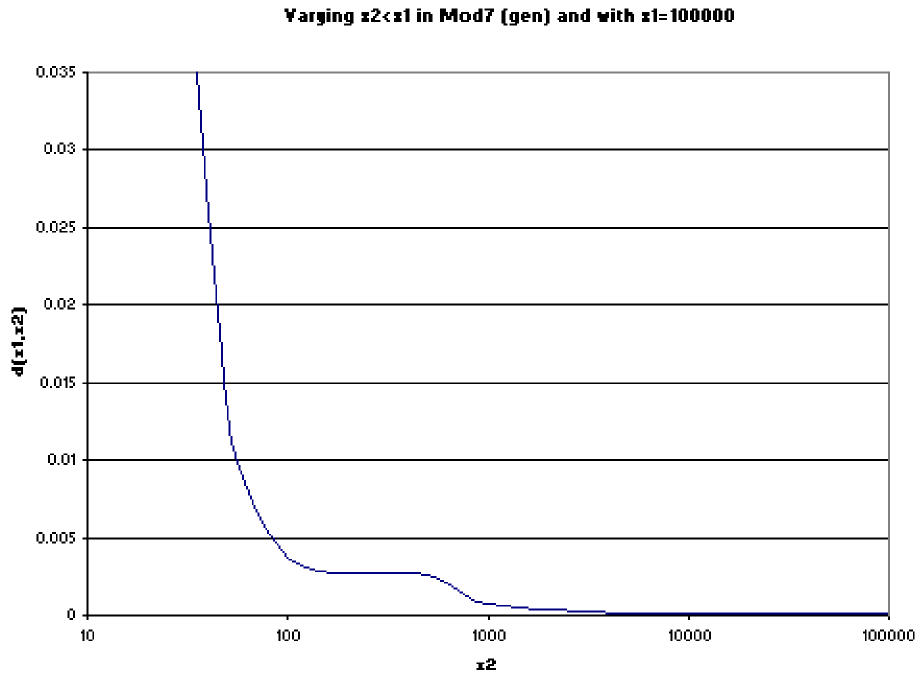

Figure 1: A rough idea (with a smoothed curve) of how abs($d(x1, x2)$) varies as $x2$ is increased, holding other variables constant. Here, the mod7 generator is used and the x-axis is a log scale.

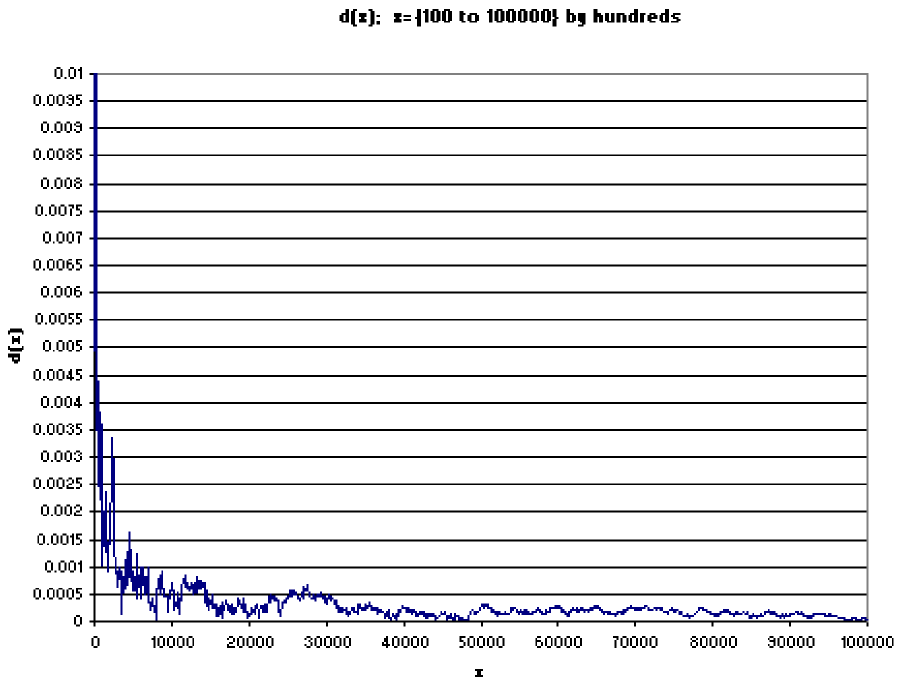

Figure 2: A much more refined picture of the sum abs($d(x)$) with the mod 7 generator. Note that the x-axis is no longer a log scale and that the y-axis is much smaller than in the previous graph. Also, unlike Graph 1, here there are 1000 sample points plotted without a smooth curve to give the best possible idea of the behaviour of the sum.