189-235A: Basic Algebra I
Assignment 1

Due: Wednesday, September 21

Let S and T be the sets $\{a, b, c\}$and $\{x, y\}$ respectively.
1. How many functions are there from S to T?
2. How many injective functions?
3. How many surjective functions?

Let X be a set, and let $\mathcal{F}(X)$ be the set of all functions from X to itself. This set is equipped with a natural binary operation $(f, g) \mapsto fg$, given by the composition of functions.
4. Show that $f(gh) = (fg)h$ for all f, g, h in $\mathcal{F}(X)$. (In other words, the operation of composition of funcions is *associative*.)
5. Show, by providing an example, that fg need not be equal to gf.
6. Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$, and that $\sum_{k=0}^{n}(-1)^k \binom{n}{k} = 0$, using the binomial theorem.

7. Using the Euclidean algorithm compute the gcd of 910091 and 3619. Show the steps in your calculation.

8. Using induction (or otherwise) show that 7 divides $8^n - 1$ for all $n \geq 0$. Use induction to show that 49 divides $8^n - 7n - 1$ for all $n \geq 0$.

9. Using induction, show that the addition law in $\mathbf{N}$ is associative directly from the axioms defining addition in $\mathbf{N}$.

10. Show that the expression

$$1^k + 2^k + 3^k + \cdots + n^k$$

can be written as a polynomial in n of degree at most $k + 1$.