189-235A: Basic Algebra I

Midterm Exam

Friday, October 22

1. Let $(u_n)_{n\geq 0}$ be the sequence of real numbers defined recursively by the rule

$$u_0 = 0, \quad u_{n+1} = 2u_n + 1.$$

Show that $u_n = 2^n - 1$ for all $n \geq 0$.

2. Compute the greatest common divisor of 121 and 77 and express the result as a linear combination of 121 and 77.

3. Solve the congruence equation $6x \equiv 10 \pmod{14}$.

4. Show that if $p \in \mathbf{Z}$ is a prime, then the ring $\mathbf{Z}_p$ of congruence classes modulo p is a field.

5. Give an example of two finite rings R_1 and R_2 which have the same cardinality but are not isomorphic. (You should justify your assertion.)

6. Show that the ring $\mathbf{C}$ of complex numbers is *not* isomorphic to the Cartesian product $\mathbf{R} \times \mathbf{R}$ of the real numbers with itself.

The next two problems are Bonus Questions

7. Let f be a polynomial in $\mathbf{Z}[x]$ of degree d and let $p \in \mathbf{Z}$ be a prime number. Show that the set

$$S = \{n \in \mathbf{Z} \text{ such that } p \text{ divides } f(n)\}$$

is the union of at most d congruence classes modulo p.

8. Let $p = 2m + 1$ be an odd prime. Show that

$$1^1 \cdot 2^2 \cdot 3^3 \cdots (p-1)^{p-1} \equiv (-1)^{[m/2]} m! \pmod{p}.$$