189-235A: Basic Algebra I
Assignment 6

Due: Wednesday, October 27

1. Show that the polynomial $x^3 - 2$ is irreducible in $\mathbf{Q}[x]$.

2. List all the monic irreducible polynomials of degree 3 in $\mathbf{Z}_2[x]$.

3. For which odd primes $p \leq 23$ is the polynomial x^2+1 irreducible in $\mathbf{Z}_p[x]$? Can you detect a pattern?

4. Find a polynomial of degree 2 in $\mathbf{Z}_6[x]$ that has four roots in $\mathbf{Z}_6$. Why does this not contradict the theorem shown in class that a polynomial in $F[x]$ of degree d has at most d roots?

5. Show that the ring $\mathbf{Z}_3[x]/(x^3 + 2x + 1)$ is a field with 27 elements.

6. Show that two polynomials $f(x)$ and $g(x)$ in $\mathbf{R}[x]$ belong to the same congruence class in $\mathbf{R}[x]/(x^2)$ if and only if $f(0) = g(0)$ and $f'(0) = g'(0)$, where $f'(x)$ is the derivative of f with respect to x.

7. Find the inverse of $[x^2 + x + 1]$ in the ring $\mathbf{Z}_2[x]/(x^3 + x + 1)$.

8. Write down all the powers of $[x]$ in the finite ring $\mathbf{Z}_2[x]/(x^3+x+1)$. What is the smallest $j > 1$ such that $[x]^j = 1$?

Bonus Questions.
9. If p is an odd prime of the form $3 + 4m$, show that the polynomial $x^2 = 1$ is irreducible in $\mathbf{Z}_p[x]$, so that $\mathbf{Z}_p[x]/(x^2 + 1)$ is a field.

10. If p is an odd prime of the form $1 + 4m$, use Wilson's Theorem to show that $a = (2m)!$ is a root in $\mathbf{Z}_p$ of the polynomial $x^2 + 1$ in $\mathbf{Z}_p[x]$.

11. If p is a prime of the form $1 + 4m$, show that the ring $\mathbf{Z}_p[x]/(x^2 + 1)$ is isomorphic to the ring $\mathbf{Z}_p \times \mathbf{Z}_p$.