THE ANALYTIC EXTENSION OF THE ZETA FUNCTION

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1. The Gamma Function

As the gamma function, the extension of the factorial function to the complex plane, is so intimately related to the zeta function, it is compulsory to start with a quick review. Now, let Γ be defined as follows:

$$\Gamma(n) = \int_0^\infty e^{-x}\, x^n \frac{dx}{x}$$

Of course this integral only converges for $Re(n) > 0$, else the integrand diverges more rapidly than $1/x$. It is simple to check the two following properties:

$$\begin{aligned}\Gamma(1) &= \int_0^\infty e^{-x}\, dx \\ &= 1\end{aligned}$$

And if $Re(n) > 1$ then we can take $u = x^{n-1}$ and $dv = e^{-x}$ and integrate by parts in the definition of $\Gamma(n)$:

$$\begin{aligned}\Gamma(n) &= 0 + (n-1) \int_0^\infty e^{-x}\, x^{n-1} \frac{dx}{x} \\ &= (n-1)\, \Gamma(n-1)\end{aligned}$$

And hence, $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}^{\geq 0}$ Now if we want to extend the gamma function to the whole complex plane, we must do a change of variable in the integrand defining the gamma function, taking $x = t^2$:

$$\Gamma(p) = 2 \int_0^\infty e^{-t^2}\, t^{2p-1}\, dt$$

If we multiply two gamma functions, expressed as above, and change to polar coordinates,

$$\begin{aligned}\Gamma(p)\,\Gamma(q) &= 4 \iint_0^\infty e^{-(t^2+s^2)}\, t^{2p-1}\, s^{2q-1}\, dt\, ds \\ &= 4 \int_0^\infty e^{-r^2}\, r^{2p+2q-1}\, dr \int_0^{\pi/2} \cos(\theta)^{2q-1}\, \sin(\theta)^{2p-1}\, d\theta \\ &= 2\Gamma(p+q) \int_0^{\pi/2} \cos(\theta)^{2q-1}\, \sin(\theta)^{2p-1}\, d\theta\end{aligned}$$

Now let $x = \sin(\theta)^2$ (hence $1 - x = \cos(\theta)^2$) we get:

$$2 \int_0^{\pi/2} \cos(\theta)^{2q-1}\, \sin(\theta)^{2p-1}\, d\theta = \int_0^1 x^{p-1}\, (1-x)^{q-1}\, dx$$

But the second integral is the definition of the beta function:

$$\begin{aligned} B(p,q) &= \int_0^1 x^p\,(1-x)^q\,\frac{dx}{x(1-x)} \\ &= \int_0^\infty \frac{y^p}{(1+y)^{p+q}}\,\frac{dy}{y} \end{aligned}$$

The second equality comes from the change $x = y/(1+y)$. Looking back, we find the famous equality:

$$B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

Knowing that

$$\int_0^\infty \frac{y^p}{(1+y)}\,\frac{dy}{y} = \frac{\pi}{\sin(\pi\,p)}$$

we can take $0 < p < 1$ and $q = 1 - p$ to get,

$$\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin(\pi\,p)} \tag{1}$$

This equation also defines $\Gamma(z)$ for values of z in all the complex plane; indeed, the right side is well defined for all z other than a real integer. As Γ is already well-defined for positive integers (1) implies that it has poles at all non positive integers, and also that it is never zero. Hence $1/\Gamma(z)$ is analytic. Furthermore we may notice that if $z \to 0$, we have

$$\begin{aligned} \Gamma(z) &= \frac{\pi}{\sin(\pi\,p)}\,\Gamma(1-z) \\ &\approx \frac{\pi}{\sin(\pi\,p)} \end{aligned}$$

So the pole of $\Gamma(z)$ at 0 is simple. In a similar fashion, one can prove that it has simple poles at the negative integers.

2. The Zeta Function

We define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

which converges for $Re(s) > 1$, since this sum is bounded by the integral $\int_0^\infty \frac{dx}{x^s}$. We might also note that, for $Re(s) > 1$, we have

$$\zeta(s) = \prod_{p\,prime} (1 - \frac{1}{p^s})^{-1} \tag{2a}$$

$$= \prod_{p\,prime} (1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots) \tag{2b}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{2c}$$

(2b) comes from the expansion of $(1 + 1/p^s)^{-1}$ in $\sum_{n \geq 0} p^{ns}$ and (2c) follows by noting that the product of those series is such that any combination of prime numbers can appear at the denominator, hence any positive integer number. In his paper, Riemann gave two proofs to the extension of the zeta function to the complex plane, and of its functional equation, both based on the following remark:

$$\int_0^\infty e^{-at}\, t^\alpha \frac{dt}{t} = \frac{1}{a^\alpha}\, \Gamma(\alpha) \tag{3}$$

Since Riemann's second proof generalize more nicely to L-funtions, although it is not as natural, and that it introduces concept more relevant to our class, we shall begin by this one.

3. The second proof

Continuing from (3), we find that,

$$\begin{aligned}\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) &= \pi^{-s/2}\,\Gamma(s/2)\,\sum_{n=1}^\infty \frac{1}{n^s}\\ &= \sum_{n=1}^\infty \frac{\pi^{-s/2}}{n^s}\,\Gamma(s/2)\\ &= \sum_{n=1}^\infty \int_0^\infty e^{-\pi\, n^2\, t}\, t^{s/2}\, \frac{dt}{t}\end{aligned}$$

$$\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) = \int_0^\infty (\sum_{n=1}^\infty e^{-\pi\, n^2\, t})\, t^{s/2}\, \frac{dt}{t} \tag{4}$$

Of course, none of the preceding equalities make sense since we do not know if the integral converges, or if we can interchange the summation and the integration order. But we might note that

$$\sum_{n=1}^\infty e^{-\pi\, n^2\, t} = \frac{1}{2}(-1 + \sum_{n=-\infty}^\infty e^{-\pi\, n^2\, t})$$

if we define

$$\theta(t) = \frac{1}{2}\sum_{n=-\infty}^\infty e^{-\pi\, n^2\, t} \tag{5a}$$

$$\Lambda(s) = \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) \tag{5b}$$

then simply note that Λ will be given by $\int_0^\infty (\theta(t) - 1/2)\, t^{s/2}\, \frac{dt}{t}$ (if it converges) which is also called a Mellin transform of $\theta(t) - 1/2$. The best way to learn about the behaviour of $\theta(t)$ is by applying Poisson's summation formula to its definition. For the time being we will simply state this theorem, but a proof can be found in the last section of this text. To apply Poisson's summation formula it is sufficient that $f(n)$ is continious with a continuous derivative f' and such that

$$\max(|f(x)|, |f'(x)|) \le c_1 \min(1, x^{-c_2})$$

for $c_1 > 0$, $c_2 > 1$ then, the serie $\sum_{n=-\infty}^{\infty} f(n)$ converges and

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \tag{6}$$

where $\hat{f}$, the Fourier transform of f, is defined. Explicitly

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(y) e^{2\pi\imath xy} dy \tag{7}$$

If we write $f_t(x) = e^{-\pi n^2 t}$ then it respects all the conditions for the summation formula and $\hat{f}_t$ is well defined by the improper integral of (7), since f decays very rapidly at $\pm\infty$. In fact,

$$\begin{aligned}
\hat{f}(x) &= \int_{-\infty}^{\infty} f_t(y) e^{2\pi\imath xy} dy \\
&= \int_{-\infty}^{\infty} e^{-\pi y^2 t + 2\pi\imath xy} dy \\
&= e^{-\pi x^2/t} \int_{-\infty}^{\infty} e^{-\pi (y\sqrt{t} - \imath x/\sqrt{t})^2} dy \\
&= \frac{e^{-\pi x^2/t}}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-\pi w^2} dw \\
&= \frac{e^{-\pi x^2/t}}{\sqrt{t}}
\end{aligned}$$

$$\hat{f}(x) = \frac{f_{1/t}(x)}{\sqrt{t}} \tag{8}$$

where we used $w = y\sqrt{t} - \imath x/\sqrt{t}$ an the fact that $\int_{-\infty}^{\infty} e^{-\pi w^2} dw = 1$

Since $\theta(t)$ is the infinte sum of $f_t(n)$ over all $n \in \mathbb{Z}$, Poisson summation formula (6) and the relation between $\hat{f}_t(x)$ and $f_{1/t}(x)$ (8) will give a functional equation for $\theta(t)$:

$$\begin{aligned}
\theta(t) &= \frac{1}{2} \sum_{n=-\infty}^{\infty} f_t(n) \\
&= \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{f_{1/t}(n)}{\sqrt{t}}
\end{aligned}$$

hence

$$\theta(t) = \frac{\theta(1/t)}{\sqrt{t}} \tag{9}$$

If we define $\phi(t) = \theta(t) - 1/2$ then $\phi(t)$ has faster than polynomial decay as $t \to \infty$ ($\theta(t) = \sum_{n\in\mathbb{Z}} f_t(n)$ has uniform converge for $t \geq a > 0$ (using Cauchy's criterion), whence the interchage of the limit $t \to \infty$ and the sum). Moreover, we get from (9)

$$\frac{1 + 2\,\phi(t)}{1 + 2\,\phi(1/t)} = \frac{1}{\sqrt{t}}$$

hence that $\phi(t) \approxeq (t^{-1/2} - 1)$ as $t \to 0^+$ (this result does not need interchange of limit and sum explicitly: it comes out of the functional equation, and the behaviour of ϕ at infinity).

Now we can properly study, the convergence of the integrals at the beginning of this section. The integral on the right side of (4) is convergent for $t \to \infty \quad \forall s$ since ϕ is of faster than polynomial decay, and as $t \to 0^+$ we get, ($\delta > 0$)

$$\begin{aligned}\int_0^\delta \phi(t)\, t^{s/2}\, \frac{dt}{t} &\approxeq \int_0^\delta (t^{-1/2} - 1)t^{s/2}\, \frac{dt}{t} \\ &\approxeq \int_0^\delta (t^{(s-3)/2} - t^{(s-2)/2})\, dt\end{aligned}$$

This last integral only converges if $Re(s) > 1$, since $\int_0^\delta t^a\, dt$ converges for only $Re(a) > -1$. As for the interchange of the integral and the sum, note that if $\int_0^\infty$ is splitted in $\int_0^d$ and $\int_d^\infty$ then in the second part the sum has uniform convergence (using Cauchy's criterion), while in the first it can be expressed as $t^{-1/2} - 1$, for d small enough. It is now legitimate to state that, for $Re(s) > 1$, (4) holds; or

$$\Lambda(s) = \int_0^\infty \phi(t)\, t^{s/2}\, \frac{dt}{t} \tag{10}$$

Furthermore, we can use the functional equation of $\phi(t)$, rewritten as $\phi(1/t) = \sqrt{t}\phi(t) + (\sqrt{t} - 1)/2$ to evaluate this integral:

$$\begin{aligned}\Lambda(s) &= \int_0^\infty \phi(t)\, t^{s/2}\, \frac{dt}{t} \\ &= -\int_1^0 \phi(t)\, t^{s/2}\, \frac{dt}{t} + \int_1^\infty \phi(t)\, t^{s/2}\, \frac{dt}{t} \\ &= \int_1^\infty \phi(1/t)\, t^{-s/2}\, \frac{dt}{t} + \int_1^\infty \phi(t)\, t^{s/2}\, \frac{dt}{t} \\ &= \int_1^\infty (\sqrt{t}\,\phi(t) + (\sqrt{t} - 1)/2)\, t^{-s/2}\, \frac{dt}{t} + \int_1^\infty \phi(t)\, t^{s/2}\, \frac{dt}{t} \\ &= \int_1^\infty \phi(t)\, t^{1-s/2}\, \frac{dt}{t} + \int_1^\infty (\sqrt{t} - 1)/2)\, t^{-s/2}\, \frac{dt}{t} + \int_1^\infty \phi(t)\, t^{s/2}\, \frac{dt}{t} \\ &= \int_1^\infty \phi(t)\, (t^{(1-s)/2} + t^{s/2})\, \frac{dt}{t} + \int_1^\infty \frac{\sqrt{t} - 1}{2}\, t^{-s/2}\, \frac{dt}{t} \\ &= I(s) + II(s)\end{aligned}$$

At this point it is wise to remark that $I(s)$ is convergent for any s, and that it is invariant under the transformation $s \to 1 - s$. Unfortunately, $II(s)$ is only convergent for $Re(s) > 1$. (In view of generalisation, another important note is that this second term comes from the fact that $\phi(t) = \theta(t) - 1/2$, and will be absent when considering more general θ functions, thus the functional equation of θ leads instantly to a functional equation for a more general Λ.) Our hope is that in evaluating $II(s)$ the constraints on s will disappear, at least partially. And indeed,

$$\begin{aligned}II(s) &= \int_1^\infty (t^{(1-s)/2} - t^{-s/2})\frac{dt}{t} \\ &= \frac{1}{2}(\frac{-2}{1-s} - \frac{2}{s}) \\ &= -\frac{1}{s(1-s)}\end{aligned}$$

since $\int_1^\infty t^{a-1}dt = 0 - 1/a$. Hence,

$$\Lambda(s) = \int_1^\infty \phi(t)\,(t^{1-s/2} + t^{s/2})\,\frac{dt}{t} - \frac{1}{s(1-s)} \tag{11}$$

Which is defined for all s, the first term is analytic as it is convergent integral of analytic functions, while the second has simple poles at 0 and 1. We can now define the meromorphic extension of $\zeta(s)$:

$$\begin{aligned}\zeta(s) &= \frac{\pi^{s/2}}{\Gamma(s/2)}\Big(\int_1^\infty \phi(t)\,(t^{1-s/2} + t^{s/2})\,\frac{dt}{t} - \frac{1}{s(1-s)}\Big)\\ &= \frac{\pi^{s/2}}{\Gamma(s/2)}\Lambda(s)\end{aligned}$$

which is meromorphic since $1/\Gamma(z)$ is analytic, and $\Lambda(z)$ has poles only at 0 and 1.

What is more important to get from (11) is the functional equation of the zeta function, which stands out simply since the right side of (11) is now completely invariant under the change $s \to 1-s$:

$$\Lambda(s) = \Lambda(1-s) \tag{12a}$$

$$\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) = \pi^{(1-s)/2}\,\Gamma((1-s)/2)\,\zeta(1-s) \tag{12b}$$

4. The first proof

It would have been more natural to continue from (3) with a more common infinite sum:

$$\begin{aligned}\Gamma(s)\zeta(s) &= \Gamma(s)\sum_{n=1}^\infty \frac{1}{n^s}\\ &= \sum_{n=1}^\infty \frac{\Gamma(s)}{n^s}\\ &= \sum_{n=1}^\infty \int_0^\infty e^{-n\,t}\,t^s\,\frac{dt}{t}\\ &= \int_0^\infty \Big(\sum_{n=1}^\infty e^{-n\,t}\Big)\,t^s\,\frac{dt}{t}\end{aligned}$$

here we use the identity $\sum_{n=1}^\infty y^n = y/(1-y)$ for $|y| < 1$, note moreover that, since this is a power serie, it has uniform convergence inside its radius.

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{t^s}{e^t - 1}\frac{dt}{t} \tag{13}$$

It easy to remark that the integral of (13) converges both at ∞ and at 0, if $Re(s) > 1$. Furthermore, the nature of the convergence of the serie also allows us to interchange the integration and summation terms. Note however that this only holds for $Re(s) > 1$. Next, consider the following contour, called a Hankel contour, denoted $H(\epsilon)$: start from a litlle above $+\infty$ in the complex plane (say $+\infty + \imath\epsilon/10$), then go toward the origin (all the while remaining above the real line) until you are at a distance ϵ from it and then start to circle counterclockwise until you have almost completed your circle, then go back to a little under

$+\infty$ (say $+\infty - \imath\epsilon/10$), this time staying a little under the real line. Consider the following integral,

$$\int_{H(\epsilon)} \frac{(-x)^s}{e^x - 1} \frac{dx}{x}$$

where $(-x)^s = e^{s\ln(-x)}$ and $\ln(r\, e^{\imath\theta}) = \ln(r) + \imath(\theta - \pi)$, that is ln is positive and real if its argument is on the negative real axis and ln is not defined on the positive real axis (which explains why we use such a "strange" contour, intuitively we are trying to "circle" the point at the origin without passing at an undefined value of ln). Then,

$$\int_{H(\epsilon)} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} = \left(\int_\infty^\epsilon + \int_{x=|\epsilon|} + \int_\epsilon^\infty\right) \frac{(-x)^s}{e^x - 1} \frac{dx}{x}$$

We can evaluate the middle part for $Re(s) > 1$, using $x = \epsilon\, e^{\imath\theta}$ (here $\theta = 0$ when $x \in \mathbb{R}^{>0}$):

$$\begin{aligned} \int_{x=|\epsilon|} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} &= \int_{x=|\epsilon|} \frac{e^{s(\ln(\epsilon)+\imath(\theta-\pi))}}{e^x - 1} \imath\, d\theta \\ &= \int_0^{2\pi} \epsilon^{s-1}\, e^{\imath(\theta-\pi)}\, \imath\, d\theta \end{aligned}$$

The second equality, taken as $\epsilon \to 0$, show us that this term tends to 0 as $\epsilon \to 0$. As for the two others:

$$\begin{aligned} \left(\int_\infty^\epsilon + \int_\epsilon^\infty\right) \frac{(-x)^s}{e^x - 1} \frac{dx}{x} &= \int_\epsilon^\infty \left(-\frac{e^{s(\ln(x)-\imath\pi)}}{e^x - 1} + \frac{e^{s(\ln(x)+\imath\pi)}}{e^x - 1}\right) \frac{dx}{x} \\ &= \left(e^{\imath\pi s} - e^{-\imath\pi s}\right) \int_\epsilon^\infty \frac{x^s}{e^x - 1} \frac{dx}{x} \\ &= 2\imath \sin(\pi s) \int_0^\infty \frac{x^s}{e^x - 1} \frac{dx}{x} \\ &= 2\imath \sin(\pi s) \Gamma(s) \zeta(s) \end{aligned}$$

Again, note that we can only take the limit $\epsilon \to 0$ if $Re(s) > 1$. From the last equalty, and using (1) we get,

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi\imath} \int_{H(\epsilon)} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} \tag{14}$$

which is an extension of the former definition, since they are equal when $Re(s) > 1$, and is meromorphic since the integral has uniform convergence on compacts, unless 0 or 1 is in them.

To get to the functional equation we must consider the domain $D(\epsilon)$ defined by the complex plane minus everything that is at a distance less than ϵ of the real positive line or any of the points $z = \pm 2n\pi\imath$, $n \in \mathbb{N}$. We can omit that this domain is not compact since if we take $D_n(\epsilon) = D(\epsilon) \cap \{z||z| < (2n+1)\pi\}$ then as $n \to \infty$ the integral on this larger circle tends to 0. Now orienting the border ∂D clockwise, we get, by Cauchy's theorem, and as ϵ tends to 0,

$$\begin{aligned} 0 &= \frac{\Gamma(1-s)}{2\pi\imath}\int_{D(\epsilon)}\frac{(-x)^s}{e^x-1}\frac{dx}{x} \\ &= -\zeta(s) - \sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{\Gamma(1-s)}{2\pi\imath}\int_{x-2\pi\imath n=|\epsilon|}\frac{(-x)^s}{e^x-1}\frac{dx}{x} \end{aligned}$$

if for each n, we choose $y = x - 2\pi\imath n$ then

$$\begin{aligned} \int_{y=|\epsilon|}\frac{(-y-2\pi\imath n)^s}{e^y-1}\frac{dy}{y+2\pi\imath n} &= -\int_{y=|\epsilon|}(-y-2\pi\imath n)^{s-1}\frac{y}{e^y-1}\frac{dy}{y} \\ &= -2\pi\imath(-2\pi\imath n)^{s-1} \end{aligned}$$

hence, going back we find

$$\begin{aligned} \zeta(s) &= -\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{\Gamma(1-s)}{2\pi\imath}(-2\pi\imath(-2\pi\imath n)^{s-1}) \\ &= \sum_{n=1}^{\infty}\Gamma(1-s)(2\pi)^{s-1}n^{s-1}(\imath^s-(-\imath)^s)/\imath \\ &= \Gamma(1-s)(2\pi)^{s-1}\sum_{n=1}^{\infty}n^{s-1}(e^{s\pi\imath/2}-e^{-s\pi\imath/2})/\imath \qquad (15) \\ &= \Gamma(1-s)(2\pi)^{s-1}2\sin(\frac{s\pi}{2})\sum_{n=1}^{\infty}\frac{1}{n^{1-s}} \\ &= \Gamma(1-s)(2\pi)^{s-1}2\sin(\frac{s\pi}{2})\zeta(1-s) \end{aligned}$$

which is another form of the funcional equation (12b), but the symetry of $s \to 1-s$ is much less in evidence here. It is easy to pass from (15) to (12b) using (1) and another property of the gamma function, namely that

$$\sqrt{\pi}\Gamma(s) = 2^{s-1}\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})$$

5. Some values of $\zeta(s)$

The zeta function is very famous for the location of its zeros, but also for the results found, but not proved, by Euler that $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, and more generally, $\zeta(2n) = \frac{(2\pi)^{2n}(-1)^{n+1}B_{2n}}{(2n)!\,2}$ where B_m is the m^{th} Bernoulli numbers. We will make a very quick survey of those subjects, begining with the latter.

The Bernoulli numbers, mentionned above, are defined as the coefficients of the power serie of $x/(e^x-1)$, explicitly

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty}\frac{B_n x^n}{n!}$$

It is easy to determine these numbers successively, though there is no computational formula to find the n^{th} one. The first ones are, $B_0 = 1$, $B_1 = -1/2$, $B_2=1/6$, $B_3 = 0$,

$B_4 = -1/30$, $B_5 = 0$, $B_6 = 1/42, \ldots$ There is a simple method of showing that the odd Bernoulli numbers are all zero except B_1: it is sufficient to show that $x/(e^x - 1) + x/2$ is an even function.

$$\begin{aligned}\frac{x}{e^x-1} + \frac{x}{2} &= \frac{x\,e^x + x}{2(e^x-1)}\\ &= \frac{x + x\,e^{-x}}{2(1-e^{-x}}\\ &= \frac{-x+(-x)e^{-x}}{2(e^{-x}-1}\\ &= \frac{-x}{e^{-x}-1} + \frac{-x}{2}\end{aligned}$$

If we take $\zeta(s)$ in the form (14) for $s = -n, n \in \mathbb{N}$, we get,

$$\begin{aligned}\zeta(-n) &= \frac{\Gamma(1+n)}{2\pi\imath}\int_{H(\epsilon)} \frac{(-x)^{-n}}{e^x-1}\frac{dx}{x}\\ &= \frac{\Gamma(1+n)}{2\pi\imath}\int_{x=|\epsilon|} \frac{(-x)^{-n}}{e^x-1}\frac{dx}{x}\end{aligned}$$

we can spare the trouble of integrating the parts $\int_\infty^\epsilon + \int_\epsilon^\infty$ as they will be multiplied by $-\sin(n\pi)\Gamma(1+n)$ and thus will be zero (when $s = +n$ the gamma function has a pole so the product gives the expected non-zero result). If we evaluate the remaining term, we get

$$\begin{aligned}\zeta(-n) &= \frac{\Gamma(1+n)}{2\pi\imath}\int_{x=|\epsilon|} \frac{(-x)^{-n}}{e^x-1}\frac{dx}{x}\\ &= \frac{n!}{2\pi\imath}\int_{x=|\epsilon|} \frac{x}{e^x-1}(-x)^{-n-1}\frac{dx}{x}\\ &= \frac{n!}{2\pi\imath}\int_{x=|\epsilon|} \sum_{k=0}^{\infty}\frac{B_k x^k}{k!}(-x)^{-n-1}\frac{dx}{x}\\ &= \frac{n!}{2\pi\imath}(-1)^{n+1}\sum_{k=0}^{\infty}\frac{B_k}{k!}\int_{x=|\epsilon|} x^{k-n-1}\frac{dx}{x}\\ &= \frac{n!}{2\pi}(-1)^{n+1}\sum_{k=0}^{\infty}\frac{B_k}{k!}\int_0^{2\pi} \epsilon^{k-n-1}e^{\imath(k-n-1)\theta}d\theta\\ &= \frac{n!}{2\pi}(-1)^{n+1}\sum_{k=0}^{\infty}\frac{B_k}{k!}2\pi\,\epsilon^{k-n-1}\delta_k^{n+1}\\ &= n!(-1)^{n+1}\frac{B_{n+1}}{(n+1)!}\\ &= \frac{(-1)^{n+1}B_{n+1}}{(n+1)}\end{aligned}$$

where δ_k^{n+1} was Kronecker's δ function ($\delta_i^j = 0$ if $i \neq j$, but if $i = j$ then $\delta_i^j = 1$).

The interchange of the serie and the integral was possible since the serie is the power serie of $x/(e^x - 1)$, which is an analytic function, so it has uniform convergence. Note that we have proved that $\zeta(-2n) = 0$, since all the B_{2n+1} are zero. Furthermore, we are one step away from proving Euler's result: if we use the functional equation of the first proof, that is (15), we get

$$\begin{aligned}\zeta(-2n+1) &= \Gamma(2n)(2\pi)^{-2n}2\sin(-(2n-1)\pi/2)\zeta(2n)\\ &= \frac{(2n-1)!2(-1)^{n+1}}{(2\pi)^{2n}}\zeta(2n)\end{aligned}$$

and since $\zeta(-2n+1) = \frac{(-1)^{2n}B_{2n}}{2n}$ we have

$$\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}B_{2n}}{(2n)!\,2}$$

The fact that the zeta function is zero at the negative even integers, can be more easily proved by using the functional equation $\Lambda(s) = \Lambda(1-s)$, indeed (12b) rewritten as

$$\zeta(s) = \frac{\sqrt{\pi}\,\Gamma((1-s)/2)\,\zeta(1-s)}{\Gamma(s/2)}$$

implies that if $s = -2n$, $1/\Gamma(s/2)$ is zero, hence the desired result.

Furthermore, we can note that except for these negative even integers, the preceding equation for $s = -z$, with $Re(z) > 0$ implies that $\zeta(-z) = 0$ if and only if $\zeta(1+z) = 0$, however if $\zeta(1+z) = 0$, and since the product (2a) converges ($Re(1+z) > 1$), then one of its term must be 0,

$$(1 - 1/p^s)^{-1} = 0 \Rightarrow p^s = 0$$

for some p, which is impossible. Hence $\zeta(s)$ has no zero outside the band $0 < Re(s) < 1$ except for the negative even integers.

It is also interesting to note that the line $s = 1/2 + \imath\, t$, $t \in \mathbb{R}$ is particular in the sense that $1 - s = \bar{s}$. Since none of the gamma functions in the functional equation vanish, the zero on that line are conjugates. Moreover, the zero of the zeta function in the strip $0 < Re(s) < 1$ are symetric around the point $s = 1/2$, as the gamma function is non vanishing on this strip.

6. The Poisson Summation Formula

Since we do not need weak hypothesis for our case, we will use strong ones: let f be a continuous function with a contimuous derivative f' such that

$$\max(|f(x)|, |f'(x)|) \leq c_1 \min(1, x^{-c_2})$$

for $c_1 > 0$, $c_2 > 1$ then its Fourier transform, $\hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{2\pi\imath\, x\, y}dx$, is defined (as the integral converges).

Furthermore if we define $F(x) = \sum_{n=-\infty}^{\infty} f(x+n)$, this sum is convergent as the integral $\int_{-\infty}^{\infty}|f(x+y)|dy$ converges for all x. Once we know it is convergent we can say it has period one (since $m = n+1$ leaves the serie invariant), and thus we can get uniform convergence on the compact $[0,1]$. Indeed, let $x \in [0,1]$ then

$$\begin{aligned} |F(x)| &= |\sum_{n=-\infty}^{\infty} f(x+n)| \\ &= \sum_{n=0}^{\infty} |f(x+n)| + \sum_{n=0}^{\infty} |f((1-x)-n)| \end{aligned}$$

It is much easier to show that both series converges since they are both bounded by $2c_1 + c_1 \sum_{n=1}^{\infty} n^{-c_2}$ which converges since $c_2 > 1$. Hence the continuity of F, and by a similar method, the existence of a continuous derivative. These are more than sufficient conditions* to have that F(x) is represented by its Fourier expansion at each x, indeed

$$F(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi \imath n x}$$

where

$$\begin{aligned} a_k &= \int_0^1 F(y) e^{-2\pi \imath k y} dy \\ &= \int_0^1 \sum_{n=-\infty}^{\infty} f(y+n) e^{-2\pi \imath k y} dy \\ &= \sum_{n=-\infty}^{\infty} \int_0^1 f(y+n) e^{-2\pi \imath k (y+n)} dy \\ &= \sum_{n=-\infty}^{\infty} \int_n^{n+1} f(w) e^{-2\pi \imath k w} dw \\ &= \int_{-\infty}^{\infty} f(w) e^{-2\pi \imath k w} dw \\ &= \hat{f}(-k) \end{aligned}$$

So if we look at F(0) we find the desired result:

$$\sum_{n=-\infty}^{\infty} f(n) = F(0) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

* Sufficient condition are that F is of period 1, piecewise continuous, of bounded variation, and such that

$$F(a) = \frac{1}{2}\left(\lim_{x \to a^+} F(x) + \lim_{x \to a^-} F(x)\right)$$

A function is said of bounded variation on a set $[a,b]$ if $\sup_P(t(P))$ is bounded, where P is a partition $\{x_i\}$, $a = x_1 < x_2 < \ldots < x_n = b$ of $[a,b]$, $t(P) = \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)|$, and the supremum is taken over all partitions of $[a,b]$.