KAM theory for dissipative systems

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Outline

- Conformally symplectic systems
- Quasi-Periodic solutions
  - A KAM Theorem
  - Small divisors
- Breakdown of analyticity
  - Dissipative Standard Map: Joint work with A. Celletti
- Breakdown of hyperbolicity
  - Dissipative Standard Map: Joint work with J.-Ll. Figueras

Conformally symplectic systems transport a symplectic form into a multiple of itself

- Gaussian thermostat

  Appear in non-equilibrium statistical mechanics (Wojtowski, Liverani, ...).
- Spin-Orbit models

  Appear in celestial mechanics (Biasco, Chierchia, Celletti, ...).
- Any two dimensional diffeomorphism

Conformally symplectic mappings

Let Ω be a symplectic form such that

$$\Omega_x(u, v) = (u, J(x)v)$$

A conformally symplectic map $f : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n$ is

$$f^*\Omega = \lambda\Omega$$

for a function $\lambda : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$.
For $n = 1$, we have that $\lambda(x) = s|det(Df(x))|^{1/2}$.
For $n \geq 2$, we have that $\lambda = constant$.

Conformally symplectic flow

Let Ω be a symplectic form such that

$$\Omega_x(u,v) = (u, J(x)v)$$

and X a flow such that there exists a funciton $\eta : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$L_X\Omega = \eta\Omega$$

The time t flow Φ_t satisfies that

$$(\Phi_t)^*\Omega = exp(\eta t)\Omega$$

Dissipative standard map

The map f_μ given by

$$\begin{aligned} y_{n+1} &= \lambda y_n + \mu + \varepsilon\, V'(x_n) \\ x_{n+1} &= x_n + y_{n+1} \end{aligned}$$

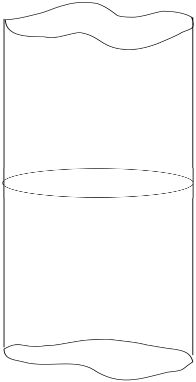
- $|Df_\mu| = \lambda$
- $\lambda = 1$, $\mu = 0$ -standard map.
- $\lambda = 0$, 1-dim map $x_{n+1} = x_n + \mu + \varepsilon V'(x_n)$.
- $\lambda = 0$, $\varepsilon = 0$, circle map $x_{n+1} = x_n + \mu$.
- $0 < \lambda < 1$ dissipative map.

Dissipative standard map for $\varepsilon = 0$

$$\omega = \frac{\mu}{1-\lambda},$$

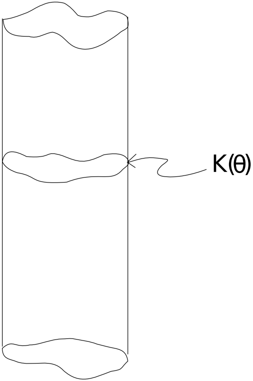
$$\omega = \frac{p}{q},$$

$$\omega \neq \frac{p}{q}.$$



Quasi-periodic solutions are orbits of the form

$$(q_n, p_n) = K(n\omega), \quad \omega \in \mathbb{R} \setminus \mathbb{Q}$$



In such a case, we have

$$f_\mu \circ K(\theta) = K(\theta + \omega) \qquad (1)$$

We will assume

$$K(\theta + 1) = K(\theta) + (1, 0)$$

“non-contractible circles”

We fix a diophantine frequecy ω

The invariance equation for $K_0 \in \mathcal{A}_\rho$, $\mu_0 \in \Lambda$

$$E[K,\mu](\theta) = f_\mu \circ K(\theta) - K(\theta+\omega)$$

A solution is K_e, μ_e so that $E[K_e,\mu_e] = 0$.
Twist and non-degeneracy

$$\mathcal{T} = \left\| \begin{pmatrix} \overline{S} & \overline{S(B_b)^0} + \overline{(M^{-1}\circ T_\omega D_\mu f_\mu \circ K)_1} \\ (\lambda-1)\mathrm{Id} & \overline{(M^{-1}\circ T_\omega D_\mu f_\mu \circ K)_2} \end{pmatrix}^{-1} \right\|$$

with $\lambda (B_b)^0(\theta) - (B_b)^0(\theta+\omega) = -(M^{-1}\circ T_\omega D_\mu f_\mu \circ K)_2$

$$M(\theta) = (DK(\theta)|J^{-1}DK(\theta)N(\theta))$$

$$N(\theta) = (DK(\theta)^T DK(\theta))^{-1}$$

An *a posteriori* theorem

Theorem (C-Celletti-de la Llave)

Fix $\omega \in \mathcal{D}_n(\nu,\tau)$, $E[K_0,\mu_0](\theta) = f_{\mu_0} \circ K_0(\theta) - K_0(\theta+\omega)$.
Suppose that $K_0 \in \mathcal{A}_\rho$, $\mu_0 \in \Lambda$, $\ell = 1$ *when* $\lambda \neq 1$, $\ell = 2$ *when* $\lambda \in [1-A, 1+A]$

$$\|E[K_0,\mu_0]\|_{\mathcal{A}_{\rho-\delta}} < C(\tau, n, \mathcal{T}, \|DK_0\|_{\mathcal{A}_0}, \|M_0^{\pm 1}\|_{\mathcal{A}_\rho}, \|N_0\|_{\mathcal{A}_\rho})\nu^{2\ell}\delta^{2\ell\tau}$$

Then there exists a $K_e \in \mathcal{A}_{\rho-2\delta}$, $\mu_e \in \Lambda$ *such that* $E[K_e,\mu_e] = 0$ *and*

$$\|K_0 - K_e\|_{\rho-2\delta} \leq C\nu^{-\ell}\delta^{-\ell\tau}\|E[K_0,\mu_0]\|_{\mathcal{A}_\rho},$$

$$|\mu_0 - \mu_e| \leq C\nu^{-\ell}\delta^{-\ell\tau}\|E[K_0,\mu_0]\|_{\mathcal{A}_\rho}.$$

Finite regularity

Theorem (C-Celletti-de la Llave)

Fix $f_\mu \in C^r$, $r > m+2$, $m > n/2 + \ell\tau + 1$
Suppose that $K_0 \in H^m$, $\mu_0 \in \Lambda$, $\ell = 1$ *when* $\lambda \neq 1$, $\ell = 2$ *when* $\lambda \in [1-A, 1+A]$

$$\|E[K_0,\mu_0]\|_{H^{m-\ell\tau}} < C(\nu, \tau, n, \mathcal{T}, \|DK_0\|_{\mathcal{A}_0}, \|M_0^{\pm 1}\|_{\mathcal{A}_\rho}, \|N_0\|_{\mathcal{A}_\rho})$$

Then there exists a $K_e \in H^{m-\ell\tau}$, $\mu_e \in \Lambda$ *such that* $E[K_e, \mu_e] = 0$ *and*

$$\|K_0 - K_e\|_{H^{m-\ell\tau}} \le C\|E[K_0,\mu_0]\|_{H^m},$$

$$|\mu_0 - \mu_e| \le C\|E[K_0,\mu_0]\|_{H^m}.$$

Consequences of *a posteriori* theorems

1. Local uniqueness and smoothness with respect to parameters
2. Numerically efficient criterion for breakdown
3. Bootstrap of regularity
4. Validation of numerical algorithms producing a solution
5. Continuation methods of numerical analysis

Note: we do not assume that the system is close to integrable

- Estimates on measure of parameters covered by quasi-periodic solutions
- Convergence of perturbative expansions

The proof of the existence theorem is based on a Nash-Moser Newton-like method.

- $E = E[K, \mu]$ is small,
- A “better” solution $(\tilde{K}, \tilde{\mu})$ satisfies $E[\tilde{K}, \tilde{\mu}] = E + DE[K, \mu](\tilde{K} - K) + D_\mu E[K, \mu](\tilde{\mu} - \mu)$ $+O(\|(\tilde{K}, \tilde{\mu}) - (K, \mu)\|^2)$,
- $\tilde{K} = K - DE^{-1}[K, \mu](E + D_\mu E[K, \mu]\sigma)$
- $\tilde{\mu} = \mu + \sigma$
- $\|E[\tilde{K}, \tilde{\mu}]\| \approx \|E\|^2$.

Nash–Moser Step

Nash–Moser:
$(DE[K,\mu])^{-1}$ is unbounded.

(bounded from one space to another, of less regular functions)

$$\tilde{K} = K - SDE[K,\mu]^{-1}(E + D_\mu E[K,\mu]\sigma)$$

S is a "smoothing" operator with restores regularity.

- By carefully adjusting the smoothing operators one can get the procedure to converge.
- One technical point is that one does not need a true inverse of $DE[K,\mu]$ but an "approximate inverse" suffices.

Example: Dissipative Standard Map

f_μ is a map.

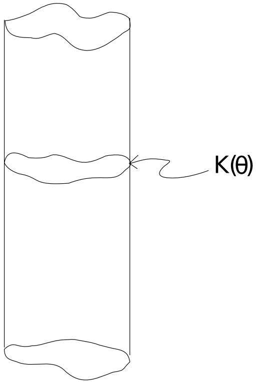
$$E(\theta) \equiv f_\mu(K(\theta)) - K(\theta + \omega)$$

$$\tilde{K} = K + \Delta \text{ and } \tilde{\mu} = \mu + \sigma$$

$$Df_\mu(K(\theta))\Delta(\theta) - \Delta(\theta + \omega) + \partial_\mu f_\mu \sigma = -E(\theta)$$

In the
integrable case we have that

$$Df_\mu \circ K = \begin{pmatrix} 1 & \lambda \\ 0 & \lambda \end{pmatrix}$$



$$\tilde{K} = K + \Delta, \quad \tilde{\mu} = \mu + \sigma$$

We have a difference equation with constant coefficients.

$$\Delta_1(\theta) - \Delta_1(\theta+\omega) = -E_1(\theta) - \lambda\Delta_2(\theta) - \sigma$$
$$\lambda\Delta_2(\theta) - \Delta_2(\theta+\omega) = -E_2(\theta) - \sigma$$

First solve equation for Δ_2 then for Δ_1

In Fourier Space the equations are simple

We have a difference equation with constant coefficients.

$$\Delta_1(\theta) - \Delta_1(\theta+\omega) = -E_1(\theta) - \lambda\Delta_2(\theta) - \sigma$$
$$\lambda\Delta_2(\theta) - \Delta_2(\theta+\omega) = -E_2(\theta) - \sigma$$

$$(1 - e^{2\pi i k\omega})\hat{\Delta}_{1,k} = -\hat{E}_{1,k} - \lambda\hat{\Delta}_{2,k}$$

$$(\lambda - e^{2\pi i k\omega})\hat{\Delta}_{2,k} = -\hat{E}_{2,k}$$

We have a difference equation with constant coefficients.

$$\Delta_1(\theta) - \Delta_1(\theta+\omega) = -E_1(\theta) - \lambda\Delta_2(\theta) - \sigma$$
$$\lambda\Delta_2(\theta) - \Delta_2(\theta+\omega) = -E_2(\theta) - \sigma$$

$$\hat{\Delta}_{1,k} = \frac{-\hat{E}_{1,k} - \lambda\hat{\Delta}_{2,k}}{1 - e^{2\pi i k\omega}}$$

$$\hat{\Delta}_{2,k} = \frac{-\hat{E}_{2,k}}{\lambda - e^{2\pi i k\omega}}$$

Small divisors for Δ_1!

and for Δ_2 if $\lambda = 1$

We define ω a Diophantine number to be

$$|\omega \cdot k - n| \geq \kappa |k|^{-\tau}.$$

In the case that we solve the constant coefficient equation

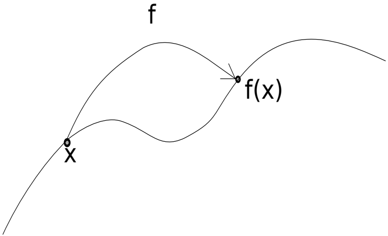
- $\lambda \neq 1$ then

$$\|\Delta_2\|_{\mathcal{A}_\rho} \leq C||\lambda| - 1|^{-1} \|E\|_{\mathcal{A}_\rho}$$

- $\lambda \in [1 - A, 1 + A]$ then

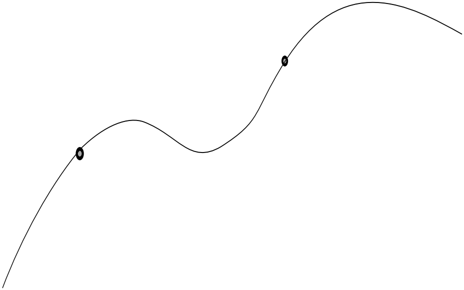
$$\|\Delta_{1,2}\|_{\mathcal{A}_{\rho-\delta}} \leq C\nu^{-1}\delta^{-\tau} \|E\|_{\mathcal{A}_\rho}$$

$$Df_\mu(K(\theta))\Delta(\theta) - \Delta(\theta+\omega) + \partial_\mu f_\mu \sigma = -E(\theta)$$



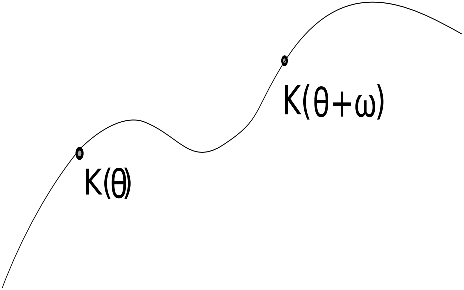
$$x \rightarrow f(x)$$

$$Df_\mu(K(\theta))\Delta(\theta) - \Delta(\theta+\omega) + \partial_\mu f_\mu \sigma = -E(\theta)$$



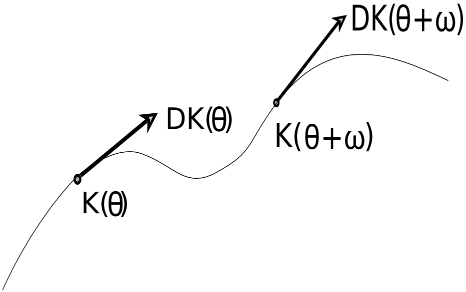
$$K(\theta) \to K(\theta+\omega)$$

$$Df_\mu(K(\theta))\Delta(\theta) - \Delta(\theta+\omega) + \partial_\mu f_\mu \sigma = -E(\theta)$$



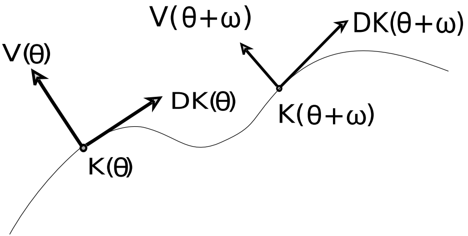
$$E(\theta) = f_\mu(K(\theta)) - K(\theta+\omega)$$

$$Df_\mu(K(\theta))\Delta(\theta) - \Delta(\theta+\omega) + \partial_\mu f_\mu \sigma = -E(\theta)$$



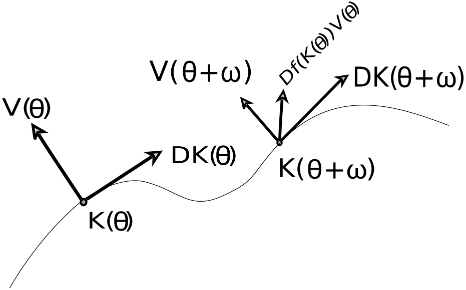
$$E(\theta) = f_\mu(K(\theta)) - K(\theta+\omega)$$
$$DE(\theta) = Df_\mu(K(\theta))DK(\theta) - DK(\theta+\omega)$$

$$Df_\mu(K(\theta))\Delta(\theta) - \Delta(\theta+\omega) + \partial_\mu f_\mu \sigma = -E(\theta)$$



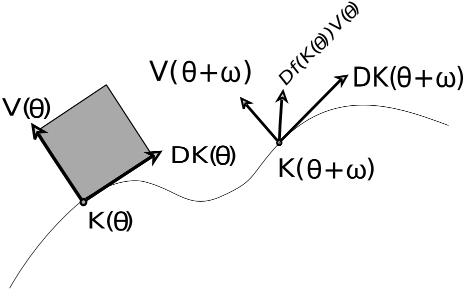
$$DE(\theta) = Df_\mu(K(\theta))DK(\theta) - DK(\theta+\omega)$$
$$V(\theta) = J^{-1}DK(\theta)N(\theta)$$
$$V(\theta+\omega) = J^{-1}DK(\theta+\omega)N(\theta+\omega)$$

$$Df_\mu(K(\theta))\Delta(\theta) - \Delta(\theta+\omega) + \partial_\mu f_\mu \sigma = -E(\theta)$$



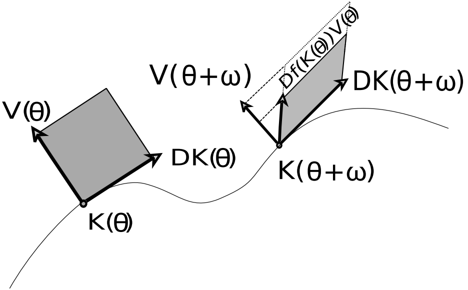
$$Df_\mu(K(\theta))V(\theta) = V(\theta+\omega)A(\theta) + DK(\theta+\omega)S(\theta)$$

$$Df_\mu(K(\theta))\Delta(\theta) - \Delta(\theta+\omega) + \partial_\mu f_\mu \sigma = -E(\theta)$$



$$Df_\mu(K(\theta))V(\theta) = V(\theta+\omega)A(\theta) + DK(\theta+\omega)S(\theta)$$

$$Df_{\mu}(K(\theta))\Delta(\theta) - \Delta(\theta+\omega) + \partial_{\mu} f_{\mu} \sigma = -E(\theta)$$



$$Df_{\mu}(K(\theta))V(\theta) = V(\theta+\omega)A(\theta) + DK(\theta+\omega)S(\theta)$$
$$A(\theta) = \lambda \mathrm{Id}$$

If

$$M(\theta) = (DK(\theta)|V(\theta))$$

then

$$Df_\mu(K(\theta))M(\theta) = M(\theta+\omega)\begin{pmatrix} I_n & S(\theta) \\ 0 & \lambda I_n \end{pmatrix}$$

In particular, if

$$\Delta(\theta) = M(\theta)W(\theta)$$

we have that

$$Df_\mu(K(\theta))\Delta(\theta) - \Delta(\theta+\omega) + \partial_\mu f_\mu \sigma = -E(\theta)$$

becomes

$$\begin{pmatrix} I_n & S(\theta) \\ 0 & \lambda I_n \end{pmatrix} W(\theta) - W(\theta+\omega) = -M^{-1}(\theta+\omega)[E[(\theta) + \partial_\mu f_\mu \sigma]$$

Algorithm for conformally symplectic maps

1. $E(\theta) = f_\mu(K(\theta)) - K(\theta + \omega)$
2. Newton's equations
   $Df_\mu(K(\theta))\Delta(\theta) - \Delta(\theta + \omega) + \partial_\mu f_\mu \sigma = -E(\theta)$
3. $M(\theta) = (DK(\theta) \quad J^{-1}(K(\theta))DK(\theta)N(\theta))$
4. $\Delta(\theta) = M(\theta)W(\theta)$
5. Solve for W from
   $$\begin{pmatrix} I_n & S(\theta) \\ 0 & \lambda I_n \end{pmatrix} W(\theta) - W(\theta+\omega) = -M^{-1}(\theta+\omega)[E(\theta) + \partial_\mu f_\mu \sigma]$$
6. Improved solution $\tilde{K}(\theta) = K(\theta) + M(\theta)W(\theta)$ and $\tilde{\mu} = \mu + \sigma$.

Quasi-Newton Method

The Quasi-Newton Step consists in using geometric identities to find an approximate solution of the linearized equation using only

- Multiplications of functions,
- Differentiations of functions,
- Solving difference equations with constant coefficients.

The same geometric cancellations above can be used to obtain **fast** and stable numerical methods.

$$O(N \log N)$$

The reduction works in any dimension
Remark that the we can solve for W from

$$W_1(\theta) - W_1(\theta+\omega) = -(M^{-1}(\theta+\omega)[E(\theta) + \partial_\mu f_\mu \sigma])_1 + S(\theta) W_2(\theta)$$
$$\lambda W_2(\theta) - W_2(\theta+\omega) = -(M^{-1}(\theta+\omega)[E(\theta) + \partial_\mu f_\mu \sigma])_2$$

both for $\lambda \neq 1$ and $\lambda = 1$.

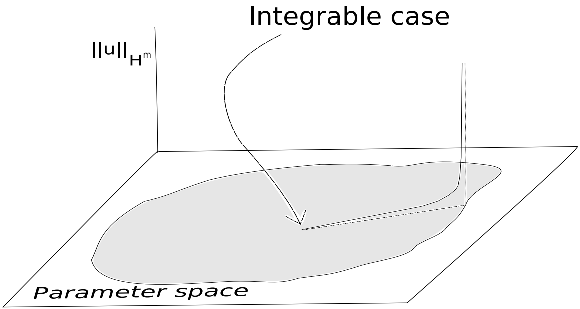
Can be solved using FFTs $O(N \log N)$ operations.

So we have an algorithm that works both in the dissipative and conservative case.

There are similar reductions for flows for flows.

In practice, the functionals we need to check are:

- Non-degeneracy of the problem
- That the approximate solution is rather regular
- Could be transformed into computer assisted proofs.



Example: the dissipative standard map

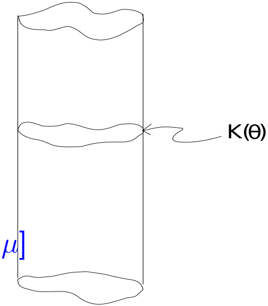
f_μ is given by

$$y_{n+1} = \lambda y_n + \mu + \varepsilon V'(x_n)$$
$$x_{n+1} = x_n + y_{n+1}$$

$$E[K, \mu] \equiv f_\mu \circ K - K \circ T_\omega$$

$$\tilde{K} = K + \Delta, \ \tilde{\mu} = \mu + \delta$$

$$Df_\mu \circ K \Delta - \Delta \circ T_\omega + \delta \partial_\mu f_\mu = -E[K, \mu]$$



$$y_{n+1} = \lambda y_n + \mu + \varepsilon\, V'(x_n)$$
$$x_{n+1} = x_n + y_{n+1}$$

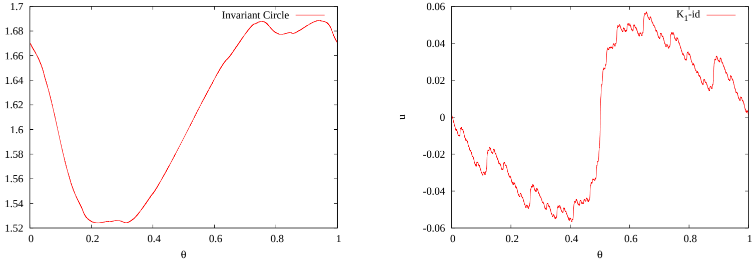


Figure: Invariant circle and conjugacy $V'(x) = \frac{\varepsilon}{2\pi} \sin(2\pi x)$ for $\varepsilon = 0.99276356$, $\lambda = 0.9$

ω is the golden mean

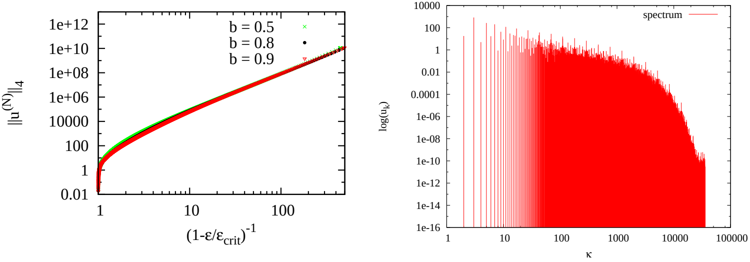


Figure: Blow-up curve and Fourier spectrum $V'(x) = \frac{\varepsilon}{2\pi}\sin(2\pi x)$ for $\varepsilon = 0.99276356$, $\lambda = 0.9$

99.89% of breakdown value

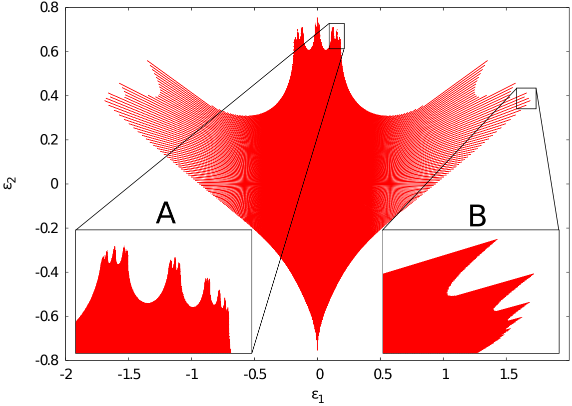


Figure: Existence domain. Parameter space $(\varepsilon_1, \varepsilon_2)$. The potential $V'(x) = \frac{\varepsilon_1}{2\pi}\sin(2\pi x) + \frac{\varepsilon_2}{4\pi}\sin(4\pi x)$ for $\lambda = 1.0$

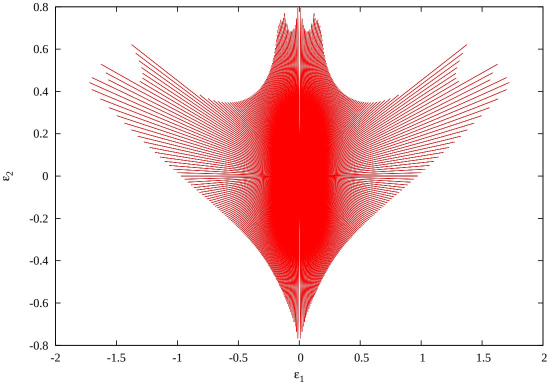


Figure: Existence domain. Parameter space $(\varepsilon_1, \varepsilon_2)$. The potential $V'(x) = \frac{\varepsilon_1}{2\pi}\sin(2\pi x) + \frac{\varepsilon_2}{4\pi}\sin(4\pi x)$ for $\lambda = 0.9$

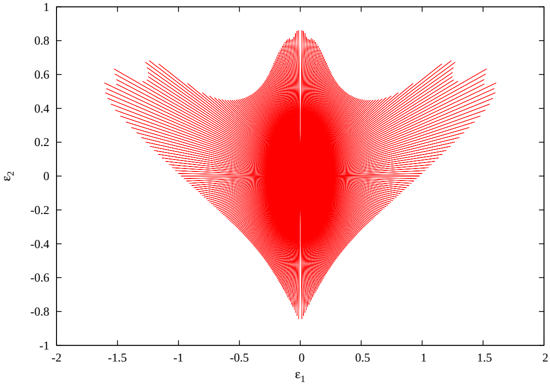


Figure: Existence domain. Parameter space $(\varepsilon_1, \varepsilon_2)$. The potential $V'(x) = \frac{\varepsilon_1}{2\pi}\sin(2\pi x) + \frac{\varepsilon_2}{4\pi}\sin(4\pi x)$ for $\lambda = 0.5$

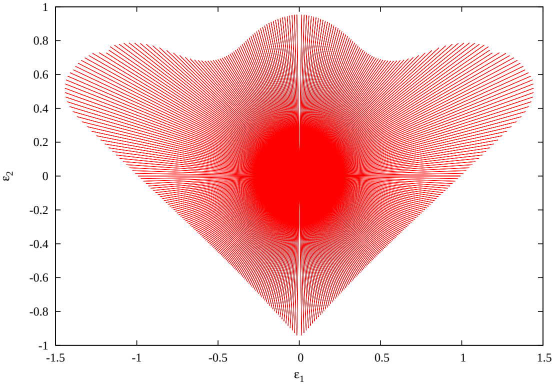


Figure: Existence domain. Parameter space $(\varepsilon_1, \varepsilon_2)$. The potential $V'(x) = \frac{\varepsilon_1}{2\pi}\sin(2\pi x) + \frac{\varepsilon_2}{4\pi}\sin(4\pi x)$ for $\lambda = 0.1$

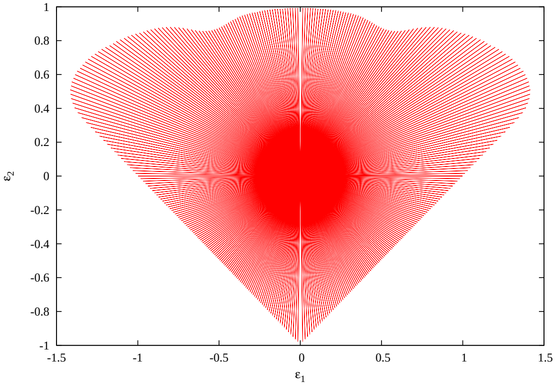


Figure: Existence domain. Parameter space $(\varepsilon_1, \varepsilon_2)$. The potential $V'(x) = \frac{\varepsilon_1}{2\pi}\sin(2\pi x) + \frac{\varepsilon_2}{4\pi}\sin(4\pi x)$ for $\lambda = 0.01$

Reducibility of the cocycle

Introduce a further change of variables

$$\tilde{M}(\theta) = M(\theta) \begin{pmatrix} \mathrm{Id} & B(\theta) \\ 0 & \mathrm{Id} \end{pmatrix}$$

So we have that

$$Df_\mu \circ K(\theta)\tilde{M}(\theta) = \tilde{M}(\theta+\omega) \begin{pmatrix} \mathrm{Id} & -\lambda B(\theta+\omega) + S(\theta) + B(\theta) \\ 0 & \lambda \mathrm{Id} \end{pmatrix}$$

Then

$$B(\theta) - \lambda B(\theta+\omega) = -S(\theta)$$

has a solution $B(\theta)$ as smooth as $S(\theta)$ for $\lambda \neq 1$ and

$$Df_\mu \circ K(\theta)\tilde{M}(\theta) = \tilde{M}(\theta+\omega) \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \lambda \mathrm{Id} \end{pmatrix}$$

In particular

We can find a frame where where the linearized dynamics is a constant matrix with n eigenvalues 1 and n eigenvalues λ.

- There exists a decomposition

$$T_{K(\theta)}\mathcal{M} = \mathrm{Range} DK(\theta) \oplus E^s_{K(\theta)}$$

- If $\lambda < 1$ the manifold $K(\mathbb{T}^n)$ is an attractor

$$(I, \varphi) \to (\lambda I, \varphi + \omega) + R(I, \phi), \quad \text{where } |R(I, \varphi)| \leq C|I|^2$$

- Lyapunov multipliers are constant $\rho_c = 1$ and $\rho_s = \lambda$. Fenichel theory $\implies$ the manifold $K(\mathbb{T}^n)$ is C^r for any $r \in \mathbb{N}$

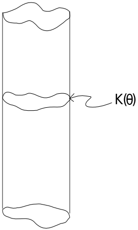
The manifold $K(\mathbb{T}^1)$ is C^r

- K is conjugation to a rotation, if $n = 1$ and $\omega \in \mathcal{D}(\nu, \tau)$ then (Herman, Sinai, Khanin, Katznelson, Ornstein,...) then K is $C^{r-(\tau-1)-\varepsilon}$
- Bootstrap of regularity $\implies$ K is analytic... and so are the decomposition into center and stable bunldes
- Then, the conjugacy K and the mapping remain analytic up to the breakdown
- Fenichel $\implies$ conjugacy cannot breakdown if manifolds are smooth
- Hyperbolicity should break down, but Lyapunov exponents are constant ($= \lambda$)

The bundles should collide

The dissipative standard map f_μ is given by

$$\begin{aligned} y_{n+1} &= \lambda y_n + \mu + \varepsilon V'(x_n) \\ x_{n+1} &= x_n + y_{n+1} \end{aligned}$$



For $\omega =$ golden mean,
we can compute the redution

$$Df_\mu(K(\theta))\tilde{M}(\theta) = \tilde{M}(\theta+\omega)\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$$

Then, the center and stable bundles are $\tilde{M}(\theta) = [W^s(\theta)|W^c(\theta)]$ and we can compute the minimum angle

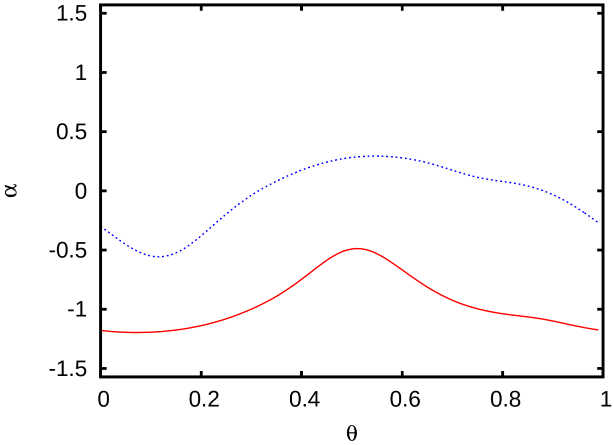


Figure: Stable and center bundle for $\lambda = 0.4$ and $\varepsilon = 0.5$

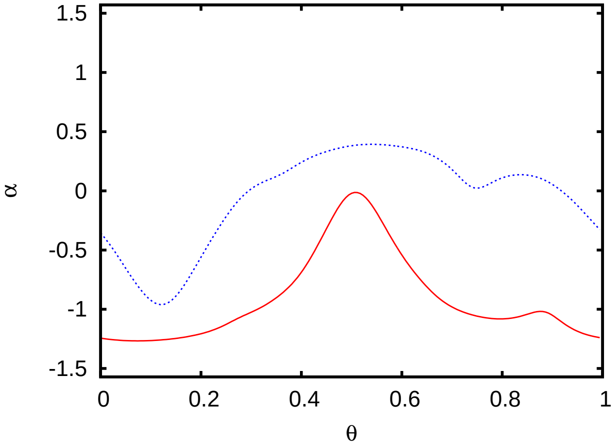


Figure: Stable and center bundle $\lambda = 0.4$ and $\varepsilon = 0.75$

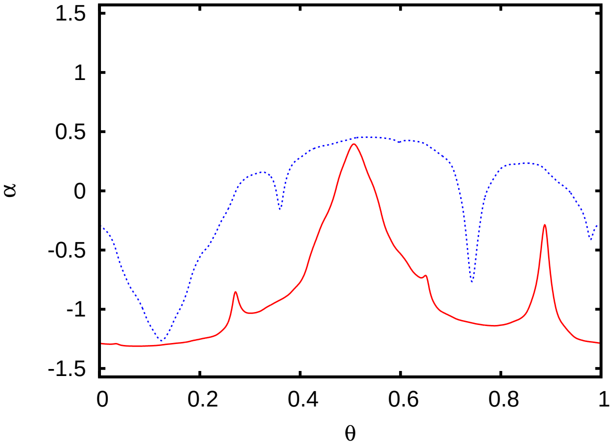


Figure: Stable and center bundle $\lambda = 0.4$ and $\varepsilon = 0.95$

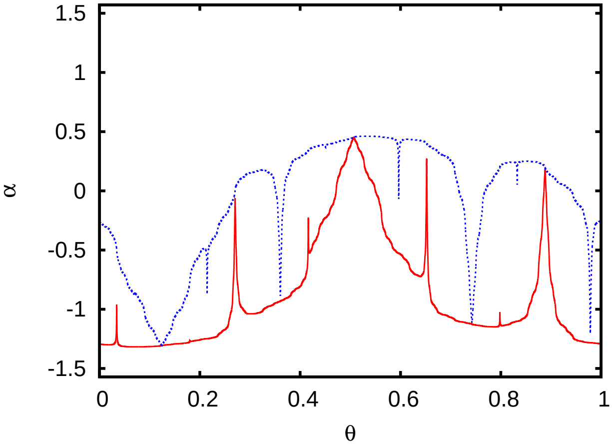


Figure: Stable and center bundle $\lambda = 0.4$ and $\varepsilon = 0.98057339$

At the breakdown value

The minimum seems to be reached always at the same point (Bjerklöv and Saprykina)

If θ is a point where the invariant bundles collide, then they collide on the orbit

$$\theta + k\omega \quad \text{for } k \in \mathbb{Z}$$

A similar phenomenon was observed in the context of the creation of Strange Nonchaotic Attractors (Jäger)

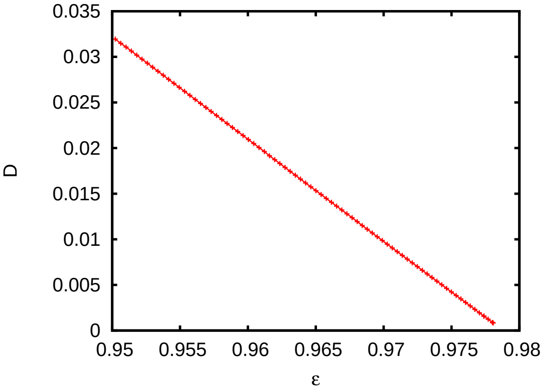


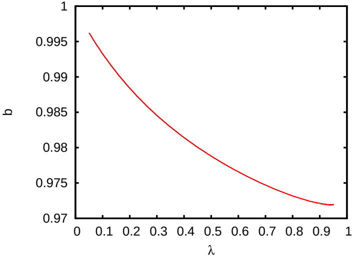
Figure: Minimum angle between the bundles with respect to ε

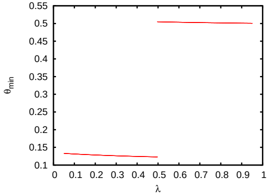
This linear asymptotics where already observed in the context of skew products

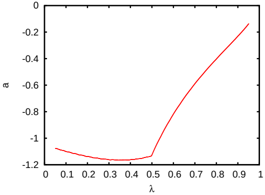
$$\begin{aligned} x' &= f(x, \theta) \\ \theta' &= \theta + \omega \end{aligned}$$

(Haro, de la Llave, Figueras, Bjerklöv, Saprykina)

$$D(\varepsilon) = a(\varepsilon - b)$$







- The a-posteriori format has many useful consequences
- Elementary geometric identites play an essential rôle in the proof of existence of quasi-periodic orbits
- Efficient and rigorously justified methods to determine the breakdown of analyticity and hyperbolicity
- The min distance between linear bundles is reached always in the same point (agrees with what Bjerklöv and Saprykina observed in Shrödinger opearators)
- Dissipative whiskered KAM tori

Thank you