

Invariant Tori in Scalar State-dependent DDEs

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ICIAM

MS89

Periodic Orbits and Tori in Applied Delay Differential Equations
Vancouver

30th June 2011

Acknowledgements

Joint Work with:

Tony Humphries(McGill), Bernd Kraukopf(Bristol)

Contributions from:

Felicia Magpantay (McGill), Finn Upham (McGill/NYU), Orianna Demasi (McGill/LBNL) Daniel Bernucci (McGill/GA Tech), Michael Snarski (McGill), Namdar Homayounfar (McGill)

Our model

$$\varepsilon \dot{u}(t) = -\gamma u(t) - \kappa_1 u(t - a_1 - c_1 u(t)) - \kappa_2 u(t - a_2 - c_2 u(t)), \quad u \in \mathbb{R}$$

- All parameters are positive

$$\gamma, \kappa_i, a_i, c_i > 0$$

- Delay equation depending on **two** delays
- The delays **depend linearly on the state**, $u(t)$
- The only non-linearity comes from the **state dependency**
- $N = 1$ studied by Mallet-Paret and Nussbaum

In Tony Humphries' talk

We observed

- Period Doubling
- Bistability
- Unimodal and multi-modal solutions as ε tends to 0 (**singular limit**)
- Solutions and delays are **bounded**...

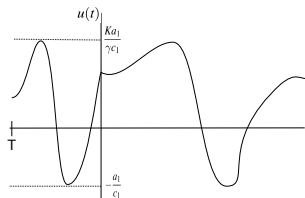
We order the delays: $0 > -\frac{a_1}{c_1} \geq \frac{-a_2}{c_2}$

Theorem (Humphries-Magpantay)

Let $\gamma > \kappa_2$, $K = \kappa_1 + \kappa_2$, and $T = \max_i \{a_1 + K \frac{c_i a_1}{\gamma c_1}\}$. If

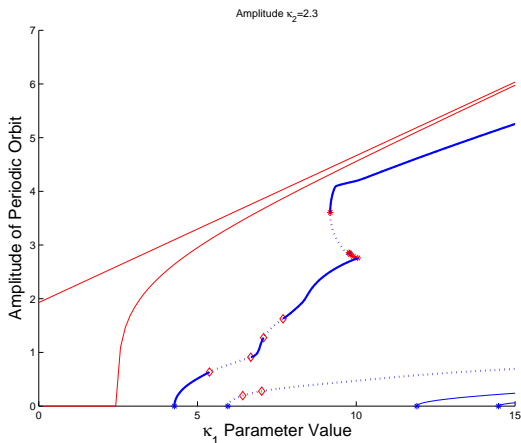
$$u(t) \in \left[-\frac{a_1}{c_1}, \frac{Ka_1}{\gamma c_1} \right] \quad \forall t \in [-T, 0], \text{ then } u(t) \in \left[-\frac{a_1}{c_1}, \frac{Ka_1}{\gamma c_1} \right], \quad \forall t \geq 0.$$

- Delays do not become advances
- The IVP is well-posed
- Solutions in this region stay in the region



$$a_2 - a_1 = 4.7$$

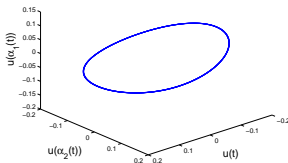
$$\kappa_2 = 2.3$$



- **Bistability** for higher values of κ_1
- A stable orbit becomes unstable (**torus bifurcation**)
- We will use a **Poincaré section** to study the dynamics around these periodic orbits

Project solutions into a three dimensional space

- Plot solutions in the space $(u(t), u(\alpha_1), u(\alpha_2))$

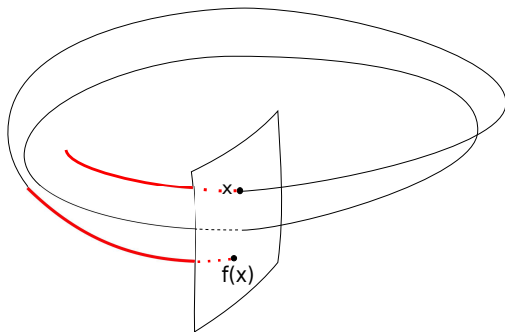


where $\alpha_1(t) = t - a_1 - c_1 u(t)$ and $\alpha_2(t) = t - a_2 - c_2 u(t)$, so

$$\varepsilon \dot{u}(t) = -\gamma u(t) - \kappa_1 u(\alpha_1(t)) - \kappa_2 u(\alpha_2(t))$$

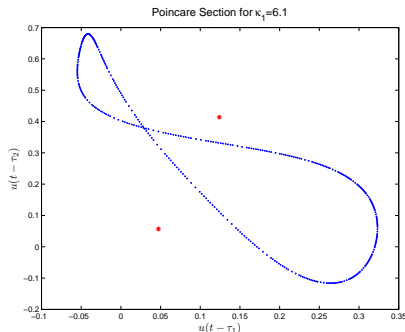
- On the Poincaré Section $u(t) = 0$ the orbits look like “points”

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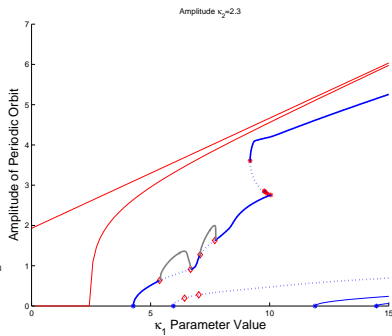
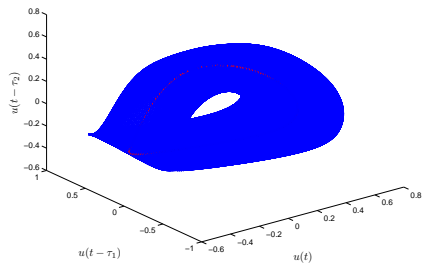
We will not show the **history function**, only the head points

On the Poincaré section



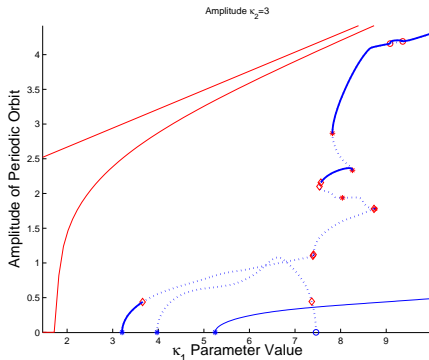
- Poor man's continuation
- Take a parameter value with no stable periodic orbit
- Integrate then IVP from unstable periodic orbit using *ddesd*
- We plot “points” on the Poincaré section

There is a stable torus



We increase κ_2

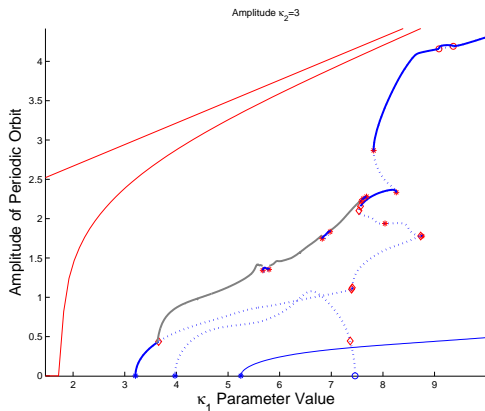
- $\kappa_2 = 3$
- Since $\kappa_2 < \gamma = 4.75$, there are bounded solutions



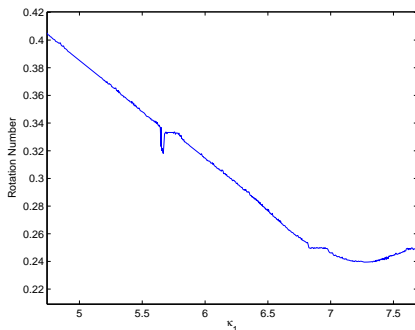
Period doubling and bistability

We increase κ_2

- $\kappa_2 = 3$
- Since $\kappa_2 < \gamma = 4.75$, there are bounded solutions

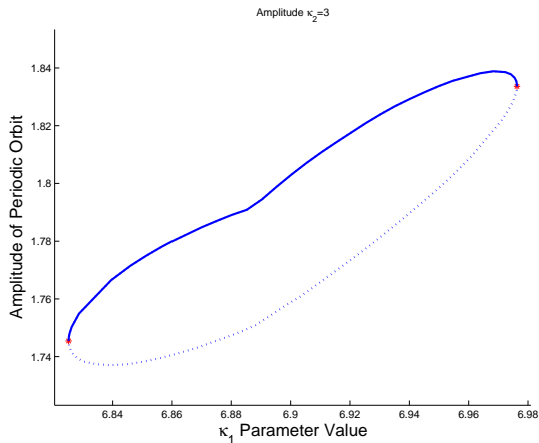


Rotation Number

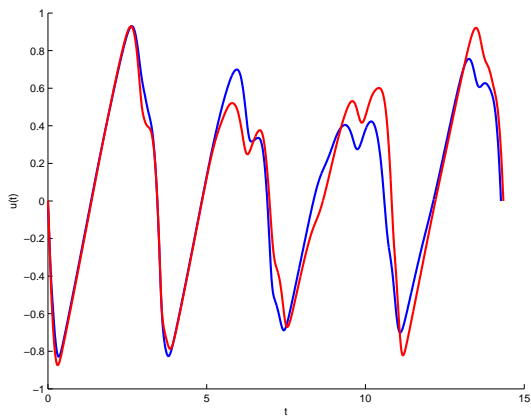


Leaves the resonance (1:4) and re-enters

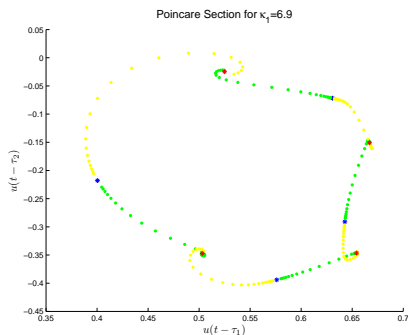
Phase locking



Phase locking

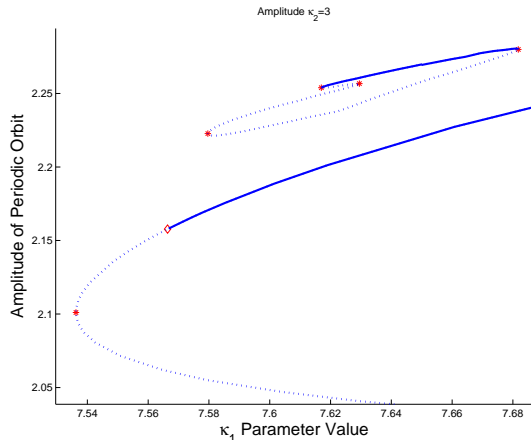


Phase locking



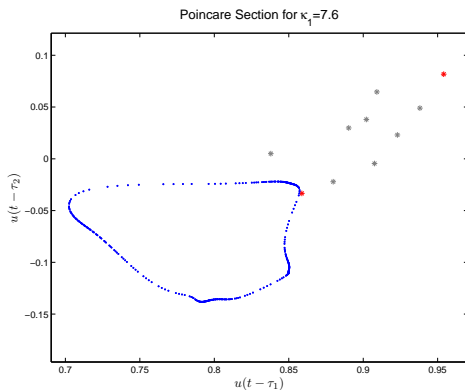
- We compute the unstable manifold from the unstable periodic orbit
- The torus has **broken up** (Normal Hyperbolicity breakdown)

Second region of phase locking



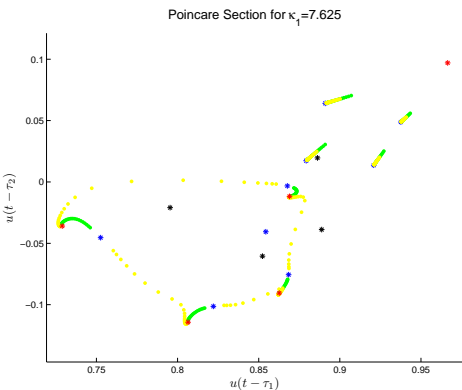
- Coexistence of **stable torus** and **stable periodic orbit**
- **Collision** of periodic orbits

Second region of phase locking



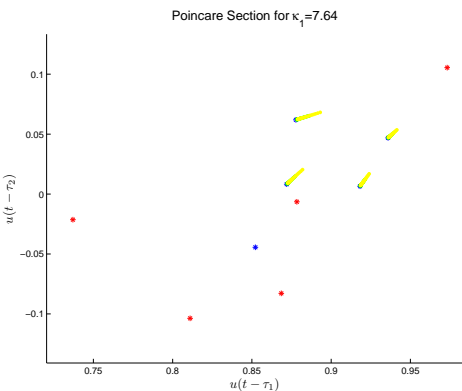
- Two period-"4" orbits created in **saddle bifurcation**
- Collision

Second region of phase locking



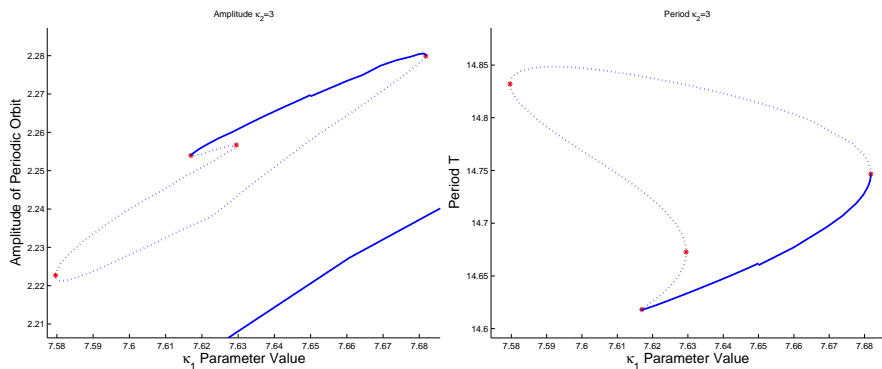
- Phase locked dynamics on 'torus'
- One period-"4" orbit is approaching unstable orbit on 'torus'

Second region of phase locking



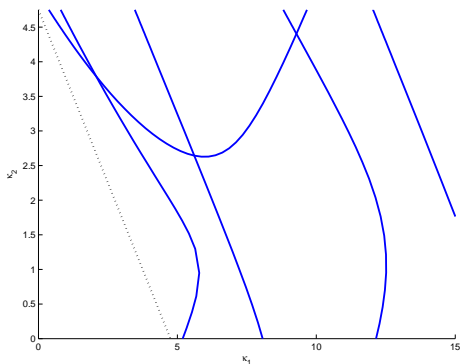
- $\kappa_1 = 7.64$
- 'torus' destroyed
- Stable 4-locked orbit persists but will be destroyed by collision

Second region of phase locking



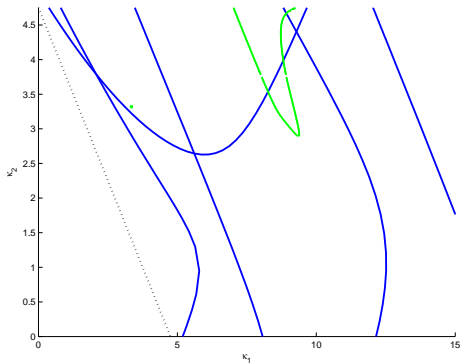
- Phase locking for the parameters $\kappa_1 \in [7.5796, 7.6818]$

Codimension-2 Bifurcations



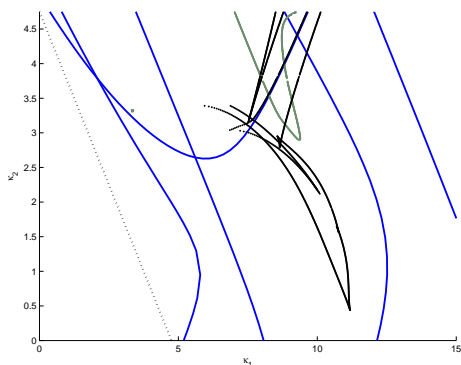
- Follow curves of Hopf bifurcations in (κ_1, κ_2) .
- Double-Hopf bifurcations

Codimension-2 Bifurcations



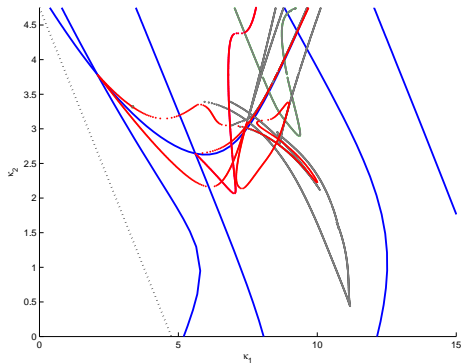
- Period-Doubling bifurcations

Codimension-2 Bifurcations



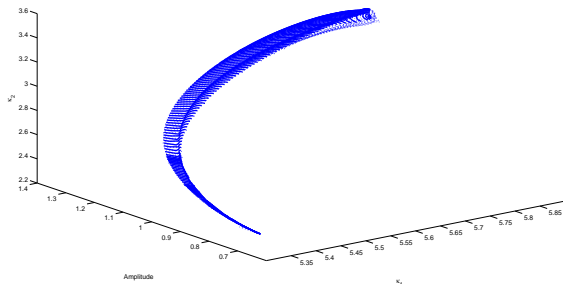
- Fold bifurcations (Saddle-node bifurcations of periodic orbits)

Codimension-2 Bifurcations



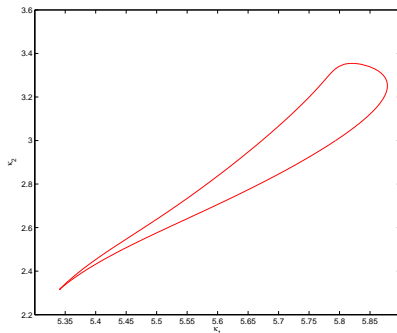
- Torus bifurcations

Codimension-2 Bifurcations



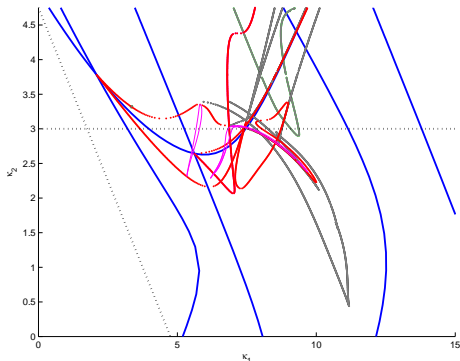
We compute the stable and unstable periodic orbits in the locking region (1:3) by continuation

Codimension-2 Bifurcations



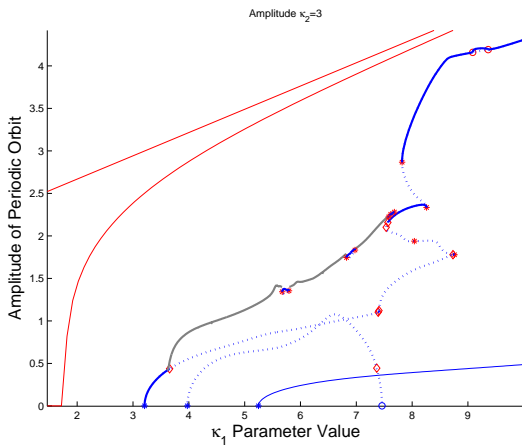
The periodic allign inside an Arnold Tongue in parameter space.

Codimension-2 Bifurcations



- Arnold tongues
- $\kappa_2 = 3$ dotted line corresponds to bifurcation diagram

$$\kappa_2 = 3$$



Conclusions

- Interesting dynamics arise from a **scalar** model with **state dependent delay**
- The only **nonlinearities** come from the **state dependency**
- We have a better algorithm for computing **unstable manifolds** on the Poincaré section
- Generalize method by Green and Krauskopf to state-dependent delays (Ongoing work with A.R. Humphries and B. Krauskopf)
- Better exploration of **Arnold Tongues** (possible overlappings)

Thank you