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Travelling waves in discrete nonlinear systems with non-nearest neighbour interactions

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Abstract
The aim of this paper is to provide a construction of travelling waves in an extended one-dimensional lattice model with non-nearest neighbour interactions. These models, coming mainly from solid state physics, are known to play an important role in the mechanisms of propagation of energy. We focus on an extended version of the Klein–Gordon chains, i.e. each particle is embedded into an an-harmonic potential and linearly coupled to their first and second neighbours. We use the technique of reduction to a finite-dimensional centre manifold to prove the existence of travelling waves of several types and investigate the role played by the interaction to second nearest neighbours.

Mathematics Subject Classification: 82C20

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The goal of this work is to study the existence of special solutions for extended lattice models. We focus here on the existence of travelling wave solutions.

The models under consideration are discrete in space and continuous in time. In particular, we focus on a non-static version of the so-called extended Frenkel–Kontorova model with non-nearest neighbour interactions described in [CdlL08b, CdlL08a]. Models with non-nearest neighbour interactions arise in materials science. For instance in [Bat06, BC99], Bates and Chmaj consider a lattice whose sites are occupied by blocks, each one consisting of many atoms arranged in a finer lattice. Each atom can exist in one of two states, A and B (such as spin up or spin down). Then, one considers that each arrangement of atoms has a potential energy of interaction consisting of (i) interactions between atoms within each block and (ii) interactions between blocks.

More precisely, we consider a chain of particles $(x_n)_{n\in\mathbb{Z}}$, each one being embedded into a nonlinear potential V, analytic in a neighbourhood of 0 (and such that $V'(0) = 0$, $V''(0) > 0$) and coupled to the first and second nearest neighbours. The equations of motion of such a system write

$$\ddot{x}_n + V'(x_n) = \gamma\Big[(x_{n+1} + x_{n-1} - 2x_n) + A(x_{n+2} + x_{n-2} - 2x_n)\Big] \qquad n \in \mathbb{Z}, \tag{1}$$

where $\gamma, A > 0$ are coupling constants.

In this paper, we focus on solutions of (1) satisfying

$$x_n(t) = \varphi(n - t/T) \tag{2}$$

for a fixed time $T \in \mathbb{R}^+$ and a smooth function φ.

A lot of literature has been devoted to discrete models like (1). In the case $A = 0$ in (1), MacKay and Aubry [MA94] construct breather solutions (i.e. spatially localized time-periodic solutions for low coupling γ). As far as propagating solutions are concerned, the case $A = 0$ has been considered in [IK00] for travelling wave solutions. One of the authors together with Guillaume James generalized the approach initiated in [IK00] to pulsating travelling waves in the works [Sir05, JS05, SJ04] for $A = 0$ in (1).

Another important discrete model is the so-called Fermi–Pasta–Ulam (FPU) lattice, i.e. a discrete model with nonlinear coupling (see [FPU55]). In [FW94], variational techniques (via concentration compactness) are used to prove the existence of travelling waves for FPU and in a series of works starting with [FP99] the authors investigate the continuum limit as well as the stability of these travelling waves.

In this work, we study the existence of small amplitude travelling waves for (1) when $A \neq 0$. Considering non-nearest neighbour interactions is reminiscent of taking into account non-local effects in the propagation of energy. It is also interesting to note that the continuous limits (at least at the formal level) depend on the sign of the coupling coefficient A. Here, we will focus on the case $A > 0$ but the computations sketched in the last section of this paper show that one can get a different behaviour when $A < 0$ is allowed. In the paper [CdlL08b], one of the authors and de la Llave considered the static version of (1), i.e.

$$V'(x_n) = \gamma[(x_{n+1} + x_{n-1} - 2x_n) + A(x_{n+2} + x_{n-2} - 2x_n)].$$

The previous system defines a four-dimensional map and reduces to a two-dimensional one when $A = 0$. The limit $A = 0$ is in some sense highly singular.

We focus here on small amplitude solutions of (1) for several parameter regimes (A, γ, T) and describe which type of solutions bifurcate from the equilibrium 0. The method of construction goes back to [IK00] and consists in reducing the system (1)–(2) to a finite-dimensional centre manifold (for some parameter regimes) which allows to capture all the dynamics of bounded small amplitude solutions. Reduction theorems can be found for instance in [CHT97, VI92] (see the appendix of this paper where the abstract reduction theorem we use is formulated). The reduction technique can be divided into three steps.

- A first step is the study of the linearized equation around 0 and particularly, a relatively precise analysis of the spectrum of the linearized operator.
- A second step ensures the solvability of an affine system related to the projection of the initial problem on the hyperbolic space.
- A third step is the study of the reduced equation on the finite-dimensional centre manifold via normal form analysis.

As previously mentioned, this strategy has been proved to be successful in the study of travelling waves in the Klein–Gordon chain (1) with $A = 0$ (see [IK00]). Our contribution consists in extending this technique to the system (1) whenever $A > 0$, involving second nearest neighbour interactions. Note that this approach via a reduction to a centre manifold has also been used to construct more complicated propagating solutions such as travelling breathers (i.e. pulsating travelling waves) in [Sir05, JS05, SJ04].

As in [IK00], in general, solitary waves (i.e. localized solutions on the lattice) do not exist. Instead, the (in some suitable sense) generic solutions are travelling waves with exponentially small periodic or quasi-periodic tails at infinity. This type of solution, which can be described as the superposition of an homoclinic (to 0) connection and a periodic (or quasi-periodic) solution, has raised a great deal of interest in recent years. Indeed, one of the main reasons for the interest in the discrete models of the type (1) is to describe the propagation of energy in spatial structures. The existence of travelling waves with exponentially small tails shows that the mechanism of propagation of energy is more involved since these generic solutions disperse energy. This phenomenon was for instance already observed in numerical computations for the Klein–Gordon lattice (see [AC98]).

2. Formulation of the problem

Recall the model under consideration

$$\ddot{x}_n + V'(x_n) = \gamma[x_{n+1} + x_{n-1} - 2x_n + A(x_{n+2} + x_{n-2} - 2x_n)] \qquad n \in \mathbb{Z}. \quad (3)$$

We consider solutions of (3) such that

$$x_n(t) = \varphi(n - t/T),$$

where $T > 0$ is a parameter (one can choose $T > 0$ since equation (3) is invariant under time reversibility). The function φ satisfies the advance-delay equation

$$\begin{aligned}\varphi''(\xi) + T^2 V'(\varphi(\xi)) = \gamma T^2\{&\varphi(\xi+1) + \varphi(\xi-1) - 2\varphi(\xi)\\ &+A(\varphi(\xi+2) + \varphi(\xi-2) - 2\varphi(\xi))\},\end{aligned} \quad (4)$$

where $\xi = n - t/T$.

Following [IK00], in order to apply centre manifold reduction, we introduce a new variable $v \in [-2, 2]$ which allows to write (4) as an evolutionary problem.

We then introduce

$$U(\xi)(v) = (\varphi(\xi), \dot{\varphi}(\xi), \Phi(v, \xi)),$$

where $\Phi(v, \xi) = \varphi(\xi + v)$. We denote δ_a the trace operators

$$\delta_a \Phi(t, v) = \Phi(t, a).$$

We end up with the following equation

$$\frac{\mathrm{d}U}{\mathrm{d}\xi} = \mathcal{L}_{A,T,\gamma} U + \mathcal{N}_T(U), \quad (5)$$

where the linear operator is given by

$$\mathcal{L}_{A,T,\gamma} = \begin{pmatrix} 0 & 1 & 0 \\ T^2[-1 - 2\gamma(1+A)] & 0 & \gamma T^2(\delta_1 + \delta_{-1}) + \gamma T^2 A(\delta_2 + \delta_{-2}) \\ 0 & 0 & \partial_v \end{pmatrix} \quad (6)$$

and the nonlinear one by

$$\mathcal{N}_T(U) = \begin{pmatrix} 0 \\ T^2(a\varphi^2 + b\varphi^3) + \text{h.o.t} \\ 0 \end{pmatrix}, \quad (7)$$

where

$$V(x) = \frac{1}{2}x^2 - \frac{a}{3}x^3 - \frac{b}{4}x^4 + \text{h.o.t.}$$

We introduce the following Banach spaces

$$\mathbb{H} = \mathbb{R}^2 \times C^0([-2, 2], \mathbb{R}) \tag{8}$$

and

$$\mathbb{D} = \left\{U \in \mathbb{R}^2 \times C^1([-2, 2], \mathbb{R}) \mid \Phi(0) = \varphi\right\}. \tag{9}$$

The operator $\mathcal{L}_{A,T,\gamma}$ maps $\mathbb{D}$ into $\mathbb{H}$ continuously and the nonlinearity $\mathcal{N}_T : \mathbb{D} \to \mathbb{D}$ is analytic (in a neighbourhood of 0) with $\mathcal{N}_T(U) = O(\|U\|_{\mathbb{D}}^2)$ as $\|U\|_{\mathbb{D}} \to 0$.

We observe that the symmetry R on $\mathbb{H}$ defined by

$$R(\varphi, \psi, \Phi) = (\varphi, -\psi, \Phi \circ z),$$

where $(\Phi \circ z)(v) = \Phi(-v)$, satisfies $(\mathcal{L}_{A,T,\gamma} + \mathcal{N}_T) \circ R = -R \circ (\mathcal{L}_{A,T,\gamma} + \mathcal{N}_T)$. Therefore, if U is a solution of the evolution equation (5) then $RU(-.)$ is also a solution, i.e. the system is reversible under R. This is reminiscent of time reversibility of equation (3).

3. Statement of main results

Our main results are described in the following theorems. We will use the following notation:

- Let Δ_0 be the set of parameters $(A, T, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ such that the centre spectrum of (6) consists of one pair of simple eigenvalues $\pm iq^*$ and a pair of double non-semisimple eigenvalues $\pm q_0$.
- Let Δ_1 be the set of parameters $(A, T, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ such that the centre spectrum of (6) consists of only one pair of simple eigenvalues $\pm iq^*$.

The existence of the two sets Δ_0 and Δ_1 is given in the next section.

Theorem 3.1 (Periodic waves). *For every $(A, T, \gamma) \in \Delta_1$ and for U near the origin in $\mathbb{D}$, the evolution equation (5) reduces to a two-dimensional reversible smooth vector field.*

Moreover, the set of solutions near 0 *of (5) constitutes a one-parameter family of periodic orbits, bifurcating from* 0.

Theorem 3.2 (Waves with exponentially small tails). *Define the quantity*

$$s_0 = \frac{T_0^2(2aK_1 + 4a^2 + 3b)}{1 - T_0^2\gamma_0(\cos(q_0) + 4A_0\cos(2q_0))} < 0,$$

where

$$K_1 = \frac{aT_0^2}{T_0^2(1 + 2\gamma_0(1 + A_0)) - 4q_0^2 - 2T_0^2\gamma_0\cos(2q_0) - 2T_0^2\gamma_0 A_0\cos(4q_0)}.$$

For every (A, T, γ) close to $(A_0, T_0, \gamma_0) \in \Delta_0$ (except for some exceptional points corresponding to strong resonances) and such that

$$s_0 < 0,$$

there exist small amplitude travelling waves of (4). These travelling waves are solutions homoclinic to exponentially small (w.r.t. the bifurcation parameter) periodic orbits.

The quantity s_0 is a coefficient of the normal form of the reduced equation on the centre manifold. It is also important to note that our analysis shows that one can expect the existence of travelling waves which are solutions homoclinic to quasi-periodic orbits and not only to periodic orbits. Indeed, for some values of the parameters, the spectral analysis shows that the centre spectrum of the linearized operator admits one pair of double non-semisimple eigenvalues together with several pairs of simple eigenvalues. In this case, the truncated (at leading order) reduced equation on the associated centre manifold admits homoclinic solutions to quasi-periodic orbits. However, in this latter case, there is no persistence result of these solutions for the untruncated equation. This situation has not been treated in [Lom00], which deals with only one pair of additional simple eigenvalues.

Remark 3.1. It has to be noticed that the method we use here can be adapted to systems involving couplings to third or even further neighbours. For the sake of simplicity, we restrict ourselves to interactions to the second neighbours.

One could also study the existence of pulsating travelling fronts for (1), i.e. solutions satisfying

$$x_n(t) = x_{n+p}(t+T),$$

for a given $p > 1$. For $A = 0$, it has been done in [JS05, Sir05] and it would be interesting to investigate the regime $A > 0$. Note that the case $p = 1$ in the previous equation corresponds to travelling waves and it is the object of this paper.

Remark 3.2. It would be interesting to investigate, at least numerically, the persistence of exponentially small tails as the parameters increase. It would also be useful to compute numerically the solutions constructed in this paper and study their properties as A goes to zero and in the negative regime for A. We postpone it to future work.

4. Spectral analysis

As described in theorem 8.1 (see the appendix), one of the main issues in the applicability of the reduction to a centre-manifold technique is to get precise information on the spectrum of the linear operator $\mathcal{L}_{A,T,\gamma}$.

From $\mathbb{D}$ into $\mathbb{H}$, the operator $\mathcal{L}_{A,T,\gamma}$ is closed with compact resolvent. Its spectrum is then discrete with isolated eigenvalues with finite multiplicity.

We consider the following eigenproblem

$$\mathcal{L}_{A,T,\gamma}\hat{U} = \sigma\hat{U},$$

where $\sigma \in \mathbb{C}$. This leads to the following dispersion relation

$$N(\sigma, T, A, \gamma) := \sigma^2 + T^2 - 2T^2\gamma(\cosh(\sigma) - 1) - 2AT^2\gamma(\cosh(2\sigma) - 1) = 0. \quad (10)$$

Simple eigenvalues of $\mathcal{L}_{A,T,\gamma}$ are simple roots of $N(\sigma, T, A, \gamma)$. Since we are mainly interested in the centre spectrum, we will consider $\sigma = \mathrm{i}q$ and $q > 0$ (note that if σ is an eigenvalue, then so is $-\sigma$). This leads to the relation

$$-q^2 + T^2 - 2T^2\gamma(\cos(q) - 1) - 2AT^2\gamma(\cos(2q) - 1) = 0. \quad (11)$$

The eigenvalue $\mathrm{i}q$ is going to be a double eigenvalue if it satisfies (11) and $(\mathrm{d}/\mathrm{d}q)N(\mathrm{i}q, T, A, \gamma) = 0$. Similarly, $\mathrm{i}q$ is going to be a triple eigenvalue if it satisfies (11), $(\mathrm{d}/\mathrm{d}q)N(\mathrm{i}q, T, A, \gamma) = 0$ and $(\mathrm{d}^2/\mathrm{d}q^2)N(\mathrm{i}q, T, A, \gamma) = 0$. For eigenvalues of multiplicity 4, the condition $(\mathrm{d}^3/\mathrm{d}q^3)N(\mathrm{i}q, T, A, \gamma) = 0$ has to be satisfied as well. Note that since we are dealing with a three-parameter problem, we do not have higher order eigenvalues. In addition, we prove in lemma 4.3 that there is no fourth order eigenvalues for the range of parameters under consideration.

In order to apply theorem 8.1 it is important to have a clear picture of the spectrum of $\mathcal{L}_{A,T,\gamma}$, at least for the ranges of parameters we are interested in. From this point of view, we focus on two types of ranges:

- First, the set of parameters for which the centre spectrum consists of only pairs of simple eigenvalues. In this case, this implies that the reduced equation on the centre manifold to have periodic and quasi-periodic solutions.
- Second, the set of parameters for which one has several pairs of simple eigenvalues together with exactly one pair of double non-semisimple eigenvalues. As we will see, this implies for the truncated (at leading order) reduced equation to have solutions homoclinic to periodic and quasi-periodic orbits. These solutions are good candidates for the travelling waves under consideration.

As described in this section, there are some ranges of parameters for which the centre spectrum does not consist of the two previously described situations. For example, additional pairs of double eigenvalues or higher order (third or fourth) eigenvalues can show up. We will always exclude these situations from our analysis but, of course, a study of the normal form (of the reduced equation) would provide the existence of small amplitude solutions bifurcating from 0 for these particular values of the parameters.

The following lemmas provide several results for the centre spectrum in the situations we are interested in. The next lemma (whose proof can be found in [IK00], lemma 1) is of general purpose and shows that the centre spectrum is well separated from the hyperbolic one.

Lemma 4.1. *For all $(T, A, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$, there exists p_0 such that all eigenvalues $\sigma = p + \mathrm{i}q$ for $p \neq 0$ satisfy $|p| \geqslant p_0$.*

We first consider simple eigenvalues. The object of the following lemma is to describe the spectrum of $\mathcal{L}_{A,T,\gamma}$ at low coupling A, time T and coupling γ being fixed.

Lemma 4.2. *For every $\gamma > 0$, there exist $A_0 > 0$ and $T_0 > 0$ such that for all $(A, T) \in (0, A_0) \times (0, T_0)$, the centre spectrum of $\mathcal{L}_{A,T,\gamma}$ consists of one pair of simple eigenvalues $\pm \mathrm{i}q^*$.*

Proof. Consider $A = 0$. Then the roots $q > 0$ of the dispersion relation $N(\mathrm{i}q, T, 0, \gamma) = 0$ are the values of q so that for a given T the following holds:

$$T^2 = \frac{q^2}{1 + 4\gamma \sin^2(q/2)}. \tag{12}$$

Now, define the function

$$f_\gamma(q) = \frac{q^2}{1 + 4\gamma \sin^2(q/2)}.$$

For every $\gamma > 0$, the function $f_\gamma(q)$ is strictly increasing on $(0, \pi)$ and therefore $1-1$ from $(0, \pi)$ into $(f_\gamma(0) = 0, f_\gamma(\pi))$. Consequently, choosing $T_0 = \sqrt{f_\gamma(\pi)}$, for all $T \in (0, T_0)$, there exists a unique q^* such that

$$q^* = f_\gamma^{-1}(T^2).$$

By applying the implicit function theorem we get the existence of a curve $A(T)$ so that $N(\mathrm{i}q^*(A, T), T, A(T), \gamma) = 0$ for T and A small enough, γ being fixed. $\square$

We now consider the case of higher order eigenvalues.

Lemma 4.3. *For every $(\gamma, T, A) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$, there are no fourth order eigenvalues.*

Proof. Fourth order eigenvalues have to satisfy

$$N(\mathrm{i}q, T, A, \gamma) = \frac{\mathrm{d}}{\mathrm{d}q}N(\mathrm{i}q, T, A, \gamma) = \frac{\mathrm{d}^2}{\mathrm{d}q^2}N(\mathrm{i}q, T, A, \gamma) = \frac{\mathrm{d}^3}{\mathrm{d}q^3}N(\mathrm{i}q, T, A, \gamma) = 0.$$

The equality $(\mathrm{d}^3/\mathrm{d}q^3)N(\mathrm{i}q, T, A, \gamma) = 0$ gives

$$\sin(q)(1 + 16A\cos(q)) = 0,$$

from which we easily rule out the case $\sin(q) = 0$. On the other hand, we have $\cos(q) = -1/16A$. The equation $(\mathrm{d}^2/\mathrm{d}q^2)N(\mathrm{i}q, T, A, \gamma) = 0$ gives the relation

$$-1 + T^2\gamma\cos(q) + 4T^2\gamma A\cos(2q) = 0.$$

Plugging $\cos(q) = -1/16A$ and using standard trigonometric formulae give the relation

$$-\frac{T^2\gamma}{32A} = 1 + 4T^2\gamma A,$$

which is impossible since $1 + 4T^2\gamma A > 0$ and $-(T^2\gamma/32A) < 0$. □

We now come to the cornerstone of the spectral analysis, i.e. the occurrence of double eigenvalues. The imaginary part q has to satisfy the two following equations:

$$N(\mathrm{i}q, T, A, \gamma) = -q^2 + T^2 - 2T^2\gamma(\cos(q) - 1) - 2AT^2\gamma(\cos(2q) - 1) = 0, \quad (13)$$

$$\frac{\mathrm{d}}{\mathrm{d}q}N(\mathrm{i}q, T, A, \gamma) = -2q + 2T^2\gamma\sin(q) + 4T^2A\gamma\sin(2q) = 0. \quad (14)$$

The occurrence of double eigenvalues is important for our purpose. Indeed, we are interested in solutions of (3) connecting 0 at $\pm\infty$ (homoclinic solutions to 0). We then have to consider bifurcations in the parameter space (T, A, γ) such that the spectrum of $\mathcal{L}_{A,T,\gamma}$ consists of a pair of weakly coupled hyperbolic eigenvalues. For some values (A, T, γ), these two pairs of hyperbolic eigenvalues are going to collide on the imaginary axis, yielding a pair of double non-semisimple eigenvalues. The next lemma locates in the parameter space such surfaces on which the spectrum has this structure.

Lemma 4.4. *Fix $\gamma > 0$ and consider the parametrized curve $\Gamma : \Omega_\gamma \mapsto \mathbb{R}^+ \times \mathbb{R}^+$ given by*

$$T_\gamma(q) = \left(\frac{q^2 - q\tan(q)}{1 + 4\gamma\sin^2(q/2) - \gamma\sin(q)\tan(q)}\right)^{1/2},$$
$$A_\gamma(q) = \frac{1}{2\gamma T_\gamma(q)^2}\left(\frac{q - T_\gamma(q)^2\gamma\sin(q)}{\sin(2q)}\right), \quad (15)$$

where Ω_γ is some subset of $\mathbb{R}^+$.

Then there exists a zero Lebesgue measure set $\tilde{\Omega}$ in $\mathbb{R}^+ \times \mathbb{R}^+$ and a subset $\Omega \subset \Gamma(\Omega_\gamma)$ such that for all $(T, A) \in \Omega\backslash\tilde{\Omega}$, the centre spectrum of $\mathcal{L}_{A,T,\gamma}$ consists of a pair of double non-semisimple eigenvalues and a finite number of pairs of simple eigenvalues.

Proof. We are looking for double eigenvalues of $\mathcal{L}_{A,T,\gamma}$ so we look for roots of (13) and (14). Solving $2\gamma T^2A$ from (14) leads to

$$2\gamma T^2A = \frac{q - T^2\sin(q)}{2\sin(q)\cos(q)}. \quad (16)$$

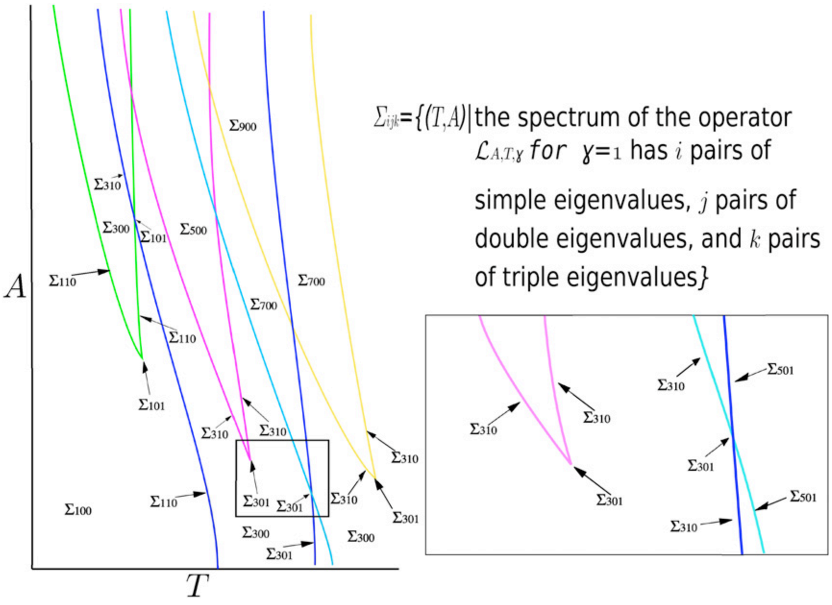

Figure 1. Curves parametrized by (15) for $\gamma = 1$ on the parameter space (T, A). The different shades represent different branches of the curves.

Plugging this expression back into equation (13), this gives an expression for T^2, namely

$$T^2 = \frac{q^2 - q\tan(q)}{1 + 4\gamma\sin^2(q/2) - \gamma\sin(q)\tan(q)}.$$

The set Ω_γ is the set of $q \in \mathbb{R}^+$ such that $T^2 > 0$ and $A > 0$. To get the expression for A we simply solve from (16).

The set $\tilde{\Omega}$ contains the values of (A, T) such that the spectrum admits triple eigenvalues and several pairs of double eigenvalues. Triple eigenvalues occur at cusp points of the curve Γ (see figure 1). We also exclude in $\tilde{\Omega}$ the curves in the parameter plane where the simple eigenvalues collide, yielding semi-simple double eigenvalues. This set is clearly negligible for the product measure in $\mathbb{R}^2$.

According to lemma 4.2, there exists a region in the parameter plane where the centre spectrum of $\mathcal{L}_{A,T,\gamma}$ consists of only one pair of simple eigenvalues. By continuity of the spectrum, we deduce that except on the set $\tilde{\Omega}$, the image of the curve Γ provides the set of parameters for which the spectrum just consists of a pair of double non-semisimple eigenvalues and a finite number of pairs of simple ones.

We refer the reader to Figure 1 where the complete spectrum is given in the parameter plane (T, A) for some $\gamma = \gamma_0$. □

In figure 2, we notice that new bifurcation curves appear when $A > 0$ that were not present in the case where $A = 0$. Now, when we fix $\gamma > 0$ and look at the bifurcation curves on the (T, A) space, the picture is slightly more involved in the sense that for every time T large enough, one can find an arbitrary small coupling $A > 0$ such that the linearized operator consists of more than one pair of simple eigenvalues. In particular, branches of double eigenvalues bifurcate from $A = 0$. These branches are present for each $\gamma > 0$ (see figure 3). More precisely, by just analysing the equation $A_\gamma(q) = 0$, one can see that they

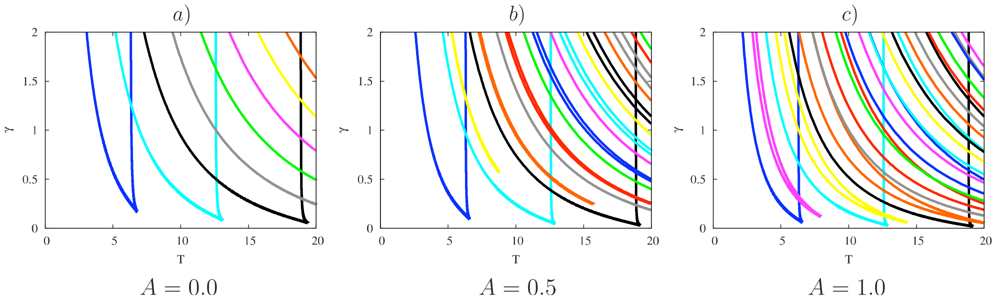

Figure 2. Parametrized curves of Γ for several values of A on the parameter space (T, γ).

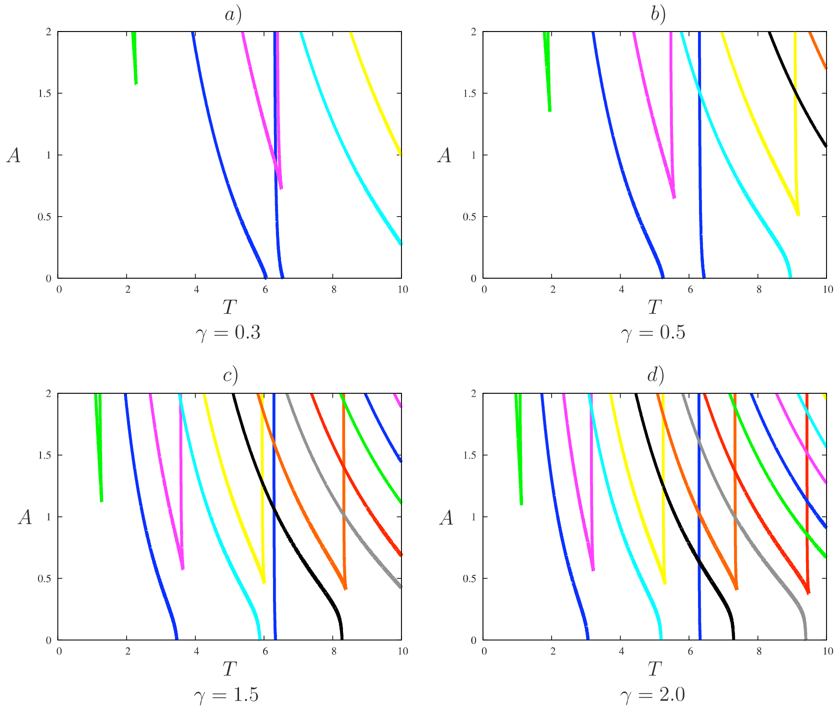

Figure 3. Curves parametrized by (15) for several values of γ on the parameter space (T, A).

occur as soon as

$$[\gamma T^2 > 1].$$

We have the following lemma, whose proof follows from a continuity argument as a consequence of the two previous lemmas. We refer the reader to figure 4 for an illustration.

Lemma 4.5. *Fix $\gamma > 0$ and A small enough. Then there exists an increasing sequence of times $\{T_n\}_{n\geqslant 0}$ such that*

- $T_0 > \sqrt{\frac{1}{\gamma}}$
- *There exists a sequence $\{\delta_n > 0\}_{n\geqslant 0}$ such that for every $T \in (T_n, T_n + \delta_n)$ the centre spectrum consists of $2n + 3$ pairs of simple eigenvalues.*

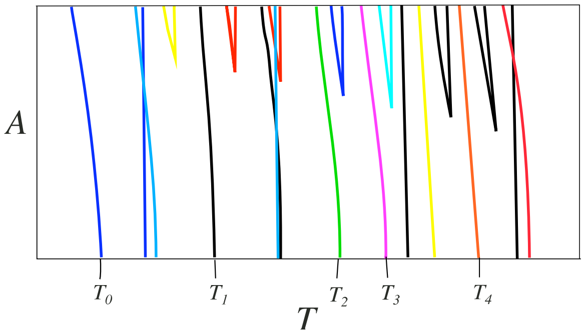

Figure 4. Curves parametrized by (15) for $\gamma = 1$ on the parameter space (T, A) for low coupling A. The sequence of times $T_0, T_1, \ldots$ illustrates the statement of lemma 4.5.

In order to study bifurcations of travelling waves near 0, we restrict ourselves to the set of A, T and γ in the parameter space such that the centre spectrum Σ_c of $\mathcal{L}_{A,T,\gamma}$ has the following structure:

$$\Sigma_c = \{\pm \mathrm{i}q_0, \pm \mathrm{i}q_1, \ldots, \pm \mathrm{i}q_p\},$$

where $\pm \mathrm{i}q_0$ is a pair of double non-semisimple eigenvalues and $\pm \mathrm{i}q_j$ $(j = 1, \ldots, p)$ are pairs of simple eigenvalues. In this case, the linear space generated by Σ_c is $(2p + 4)$-dimensional.

As previously mentioned, the reason why we consider this type of bifurcation is that the pair of double non-semisimple eigenvalues gives rise to homoclinic connections for the normal form of the evolution equation (5) on the centre manifold. The additional pairs of simple eigenvalues will lead to oscillatory tails.

One can check the following properties.

Lemma 4.6. *Let (A, γ, T) be such that the spectrum of $\mathcal{L}_{A,T,\gamma}$ is Σ_c and let V_0, V_j for $j = 1, \ldots, p$ be the eigenvectors associated with $\mathrm{i}q_0$ and $\mathrm{i}q_j$, respectively. Denote by $\hat{V}_0$ the generalized eigenvector associated with $\mathrm{i}q_0$. The eigenvectors can be chosen in the following way:*

$$V_0 = \begin{pmatrix} 1 \\ \mathrm{i}q_0 \\ \mathrm{e}^{\mathrm{i}q_0 v} \end{pmatrix}, \qquad \hat{V}_0 = \begin{pmatrix} 0 \\ 1 \\ v\mathrm{e}^{\mathrm{i}q_0 v} \end{pmatrix} \qquad \textit{and} \quad V_j = \begin{pmatrix} 1 \\ \mathrm{i}q_j \\ \mathrm{e}^{\mathrm{i}q_j v} \end{pmatrix}$$

for $j = 1, \ldots, p$. Moreover, these eigenvectors satisfy

$$RV_0 = \bar{V}_0, \qquad RV_j = \bar{V}_j, \qquad R\hat{V}_0 = -\bar{\hat{V}}_0.$$

5. Problem on the hyperbolic subspace and reduction to a centre manifold

In this section we compute the spectral projection on the hyperbolic subspace (invariant subspace under $\mathcal{L}_{A,T,\gamma}$ corresponding to the hyperbolic spectral part) and prove a regularity result for the associated inhomogeneous linearized equation. This result is a crucial assumption for applying centre manifold reduction theory (see theorem 8.1 in the appendix). Our proof closely follows the method given in [IK00]. For the sake of completeness, we give the proof in this case, making sure that the additional coupling term in A is not an obstruction to the solvability.

We call P_0, $P_1, \ldots, P_p$, respectively, the spectral projection on the four-dimensional invariant subspace associated with $\pm \mathrm{i}q_0$ and on the p two-dimensional subspaces corresponding to $\pm iq_j$, for $j = 1, \ldots, p$. We also define $P = \sum_{j=0}^{p} P_j$ (spectral projection on the $(2p+4)$-dimensional central subspace) and use the notations $\mathbb{D}_h = (\mathbb{I} - P)\mathbb{D}$, $\mathbb{H}_h = (\mathbb{I} - P)\mathbb{H}$, $\mathbb{D}_c = P\mathbb{D}$, $U_h = (\mathbb{I} - P)U$. The affine linearized system on $\mathbb{H}_h$ reads

$$\frac{\mathrm{d}U_h}{\mathrm{d}\xi} = \mathcal{L}_{A,T,\gamma} U_h + F_h(\xi), \tag{17}$$

where $F(\xi) = (0, f(\xi), 0)^T$ lies in the range of the nonlinear operator (7). We shall note $U_h = (\varphi^h, \dot{\varphi}^h, \Phi^h(v))^T$.

Our aim is to check the regularity property of equation (17) (see [VI92], property (ii) p 127 or see the appendix). This property can be stated as follows. For a given Banach space Z and $\alpha \in \mathbb{R}^+$, we introduce the following Banach space:

$$E_j^\alpha(Z) = \left\{ f \in C^j(\mathbb{R}, Z) \; \|f\|_j = \max_{0 \leqslant k \leqslant j} \sup_{t \in \mathbb{R}} e^{-\alpha|t|} |D^k f(t)| < \infty \right\}. \tag{18}$$

We have to prove that system (17) admits a unique solution U_h in $E_0^\alpha(\mathbb{D}_h) \bigcap E_1^\alpha(\mathbb{H}_h)$ for $0 \leqslant \alpha < \alpha_0$ (for some $\alpha_0 > 0$), the operator $K_h : E_0^\alpha(\mathbb{R}) \to E_0^\alpha(\mathbb{D}_h)$, $f \mapsto U_h$ being bounded.

We do not know *a priori* if the operator $\mathcal{L}_{A,T,\gamma}$ is sectorial, providing explicit estimates on U_h. Following the method in [IK00], we compute by hand the solution U_h and provide necessary estimates.

5.1. Computation of the spectral projection on the hyperbolic subspace

The spectral projection on the central subspace is defined by the Dunford integral

$$P = \frac{1}{2i\pi} \int_C (\sigma \mathbb{I} - \mathcal{L}_{A,T,\gamma})^{-1} \, \mathrm{d}C, \tag{19}$$

where C is a regular curve surrounding $\pm iq_0, \pm iq_1, \ldots, \pm iq_p$. The spectral projection on the hyperbolic subspace is $P_h = \mathbb{I} - P$. We shall use the following result for computing P_h.

Lemma 5.1. *Let $h(z) = f(z)/g(z)$ be a function of $z \in \mathbb{C}$. Assume the function $f(z)$ is differentiable at $z = z_0$ and the function $g(z)$ admits a double zero at $z = z_0$. Then the residue of h at $z = z_0$ is given by*

$$\mathrm{Res}(h, z_0) = \frac{2f'(z_0)g''(z_0) - \frac{2}{3} f(z_0)g'''(z_0)}{g''(z_0)^2}. \tag{20}$$

Proof. Since g has a zero of order two, there exists a function φ defined on a neighbourhood of z_0 such that $g(z) = (z - z_0)^2 \varphi(z)$, with $\varphi(z_0) \neq 0$, $\varphi'(z_0) \neq 0$.

Then the residue will be given by the following limit

$$\mathrm{Res}(h, z_0) = \lim_{z \to z_0} ((z - z_0)^2 h(z))' = \lim_{z \to z_0} \frac{f'(z)\varphi(z) - f(z)\varphi'(z)}{\varphi^2(z)}.$$

We obtain the result by computing the limit. □

In the following lemma, we compute the spectral projection on the hyperbolic subspace of a vector F lying in the range of the nonlinear operator (7).

Lemma 5.2. *Let $F \in \mathbb{D}$ be a vector of the type $F = (0, f, 0)^T$. Then the projection of F on the hyperbolic subspace reads*

$$F_h = (0, k_1 f, k_2(v) f)^T, \tag{21}$$

where $k_1 \in \mathbb{R}$ and $k_2 \in C^\infty([-2, 2])$ depend on A, γ and T.

Proof. We first compute the resolvent of $\mathcal{L}_{A,T,\gamma}$. One has to solve $(\sigma\mathbb{I} - \mathcal{L}_{A,T,\gamma})U = F$ for $U = (u_1, u_2, u_3)^T$, which yields the system

$$\sigma u_1 = u_2, \tag{22}$$

$$\beta_1 u_1 - \gamma T^2[(u_3(1) + u_3(-1)) + A(u_3(2) + u_3(-2))] = f, \tag{23}$$

$$u_3 = u_1 e^{\sigma v} \tag{24}$$

for $\beta_1 = T^2(1 + 2\gamma(1 + A)) + \sigma^2$. We have then

$$u_1 = \frac{f}{N(\sigma, T, A, \gamma)}.$$

Now we compute the spectral projection P_j. Since $\sigma_j = \mathrm{i}q_j$, for $j = 1, \ldots, p$, are simple roots of the dispersion relation, one has

$$\mathrm{Res}(u_1, iq_j) = \frac{f}{-2q_j + 2\gamma T^2 \sin(q_j) + 4A\gamma T^2 \sin(2q_j)}.$$

Denoting $(P_1 F)_i$ the ith component of $P_j F$, we get consequently

$$(P_j F)_1 = \mathrm{Res}(u_1, \mathrm{i}q_j) + \mathrm{Res}(u_1, -\mathrm{i}q_j) = 0.$$

In the same spirit

$$(P_j F)_2 = \frac{-2q_j f}{-2q_j + 2\gamma T^2 \sin(q_j) + 4A\gamma T^2 \sin(2q_j)},$$

$$(P_j F)_3 = \frac{2\cos(q_j v) f}{-2q_j + 2\gamma T^2 \sin(q_j) + 4A\gamma T^2 \sin(2q_j)},$$

which completes the computation of the projections $P_j F$. For computing the spectral projection P_0 associated with the double eigenvalues $\pm\mathrm{i}q_0$, we use formula (20). These computations lead to equation (21). □

5.2. *Resolution of the affine equation for bounded functions of t*

We first solve (17) in the spaces E_j^α with $\alpha = 0$, i.e. we consider bounded functions of t (note that $E_j^0(\mathbb{H}) = C_b^j(\mathbb{H})$). Fixing $\alpha = 0$ will allow us to take the Fourier transform in time of the system in the space of tempered distributions $S'(\mathbb{R})$.

From (17), we directly deduce

$$\Phi^h(t, v) = \varphi^h(t + v) + \int_0^v k_2(s) f(t + v - s)\,\mathrm{d}s \tag{25}$$
$$= \varphi^h(t + v) + \int_t^{t+v} k_2(t + v - s) f(s)\,\mathrm{d}s,$$

(this expression comes from the last two equations of the affine linear system and from conditions $\Phi(0, t) = \varphi(t)$). From the previous equations and the fact that k_2 and its derivatives are bounded functions of v, we deduce that

$$\|\Phi^h\|_{E_0^0(C^1[-2,2])} \leqslant \|\varphi^h\|_{E_1^0} + C\|f\|_{E_0^0}. \tag{26}$$

We now have to estimate φ^h and $\dot{\varphi}^h$. Taking the Fourier transform in time of the system (17) in the tempered distributional space $S'(\mathbb{R})$, we have

$$(\mathrm{i}k - \mathcal{L}_{A,T,\gamma})\hat{U}_h = \hat{F}_h. \tag{27}$$

We deduce

$$\hat{\varphi}^{h} = \mathrm{i}k\hat{\varphi}^{h},$$
$$\hat{\varphi}^{h} = \mathrm{e}^{\mathrm{i}kv}\hat{\varphi}^{h} + \hat{f}\int_0^v \mathrm{e}^{\mathrm{i}k(v-s)}k(s)\,\mathrm{d}s.$$

For $\hat{\varphi}^h$, we have

$$N(\mathrm{i}k, T, A, \gamma)\hat{\varphi}^h = \hat{f}. \tag{28}$$

Then from (28) we define the function $\hat{G}$ as

$$[\hat{\varphi}^h = \hat{G}\hat{f}].$$

The operator $(\mathrm{i}k - \mathcal{L}^h_{A,T,\gamma})^{-1}$ (the operator $\mathcal{L}^h_{A,T,\gamma}$ denoting the restriction of $\mathcal{L}_{A,T,\gamma}$ to the hyperbolic space) is analytic in a strip around the real axis. We deduce that $\hat{G}$ is an analytic function in the same strip. Moreover, $\hat{G}$ behaves like $O(1/k^2)$ as $k \to \pm\infty$ due to the fact that $N(\mathrm{i}k, A, T, \gamma) = O(k^2)$ as $k \to \pm\infty$. The fact that the dispersion relation behaves like k^2 is important in our context. Here, even if the coupling term is more complicated, involving the second nearest neighbours, the perturbation is still $O(1)$ in terms of k.

Since $N(\mathrm{i}q_j, A, T, \gamma) = 0$, $N'(\mathrm{i}q_0, A, T, \gamma) = 0$ and $N'(\mathrm{i}q_j, A, T, \gamma)$ and $N''(\mathrm{i}q_0, A, T, \gamma)$ do not vanish, equation (28) yields

$$\hat{\varphi^h} = \hat{G}\hat{f} + b_0^+\delta'_{\mathrm{i}q_0} + b_0^-\delta'_{-\mathrm{i}q_0} + \sum_{j=1}^{p}\left(a_j^+\delta_{\mathrm{i}q_j} + a_j^-\delta_{-\mathrm{i}q_j}\right). \tag{29}$$

Furthermore, $k \to (1+|k|^2)^{1/2}\hat{G}$ belongs to $L^2(\mathbb{R})$. Therefore, using the inverse Fourier transform and lemma 3 p448 of [IK00], there exists a function $G \in H^1_\delta(\mathbb{R})$ (i.e. $\mathrm{e}^{\delta|t|}G \in H^1(\mathbb{R})$, $\delta > 0$ small enough) such that $\hat{G}$ is the unique Fourier transform of G. We have the following estimates

$$\left\|\frac{\mathrm{d}G}{\mathrm{d}t} * f\right\|_{C_b^0} = \sup_{t\in\mathbb{R}}\left|\int_{\mathbb{R}}\frac{\mathrm{d}G}{\mathrm{d}t}(t-s)f(s)\,\mathrm{d}s\right| \leqslant C(\delta)\|f\|_{C_b^0}\|G\|_{H^1_\delta(\mathbb{R})}. \tag{30}$$

Now we make the solution of (17) explicit. We set $\tilde{U}_h = (\tilde{u}_1^h, \tilde{u}_2^h, , \tilde{u}_3^h)^T$ and

$$\tilde{u}_1^h = G * f,$$
$$\tilde{u}_2^h = \frac{\mathrm{d}\tilde{u}_1^h}{\mathrm{d}t},$$
$$\tilde{u}_3^h(t, v) = \tilde{u}_1^h(t+v) + \int_0^v k_2(s)f(t+v-s) + k_8(s)\,\mathrm{d}s.$$

By construction, $\tilde{u}^h$ satisfies (17) and $P\hat{\tilde{U}}_h = 0$ (hence $P\tilde{U}_h = 0$) for $f \in E_0^\alpha(\mathbb{R})$ with $\alpha < 0$ ($\hat{f}_i$ are analytic functions in a strip around the real axis). Since the computations are formally the same for $\alpha = 0$, we have $P\hat{\tilde{U}}_h = 0$ for $\alpha = 0$, hence $P\tilde{U}_h = 0$ for $\alpha = 0$. Moreover, we have the estimate

$$\|\tilde{U}_h\|_{C_b^0(\mathbb{D}_h)\bigcap C_b^1(\mathbb{H}_h)} \leqslant C\|f\|_{C_b^0(\mathbb{R})} \tag{31}$$

due to estimates (26), (30) (with analogous estimates on H_2). For $\alpha = 0$, we obtain u^h by adding to $\tilde{u}^h$ the inverse Fourier transforms of Dirac measures, i.e.

$$u_1^h = \tilde{u}_1^h + \sum_{j=1}^{p}(a_j^+\mathrm{e}^{\mathrm{i}q_1t} + a_j^-\mathrm{e}^{-\mathrm{i}q_1t}) + (a_0^+ + \mathrm{i}tb_0^+)\mathrm{e}^{\mathrm{i}q_0t} + (a_0^- - \mathrm{i}tb_0^-)\mathrm{e}^{-\mathrm{i}q_0t}, \tag{32}$$

$$u_2^h = \tilde{u}_2^h + \sum_{j=1}^{p} (c_j^+ \mathrm{e}^{\mathrm{i}q_j t} + c_j^- \mathrm{e}^{-\mathrm{i}q_j t}) + (c_0^+ + \mathrm{i}td_0^+)\mathrm{e}^{\mathrm{i}q_0 t} + (c_0^- - \mathrm{i}td_0^-)\mathrm{e}^{-\mathrm{i}q_0 t}. \quad (33)$$

Since $P\tilde{U}_h = 0$, we have $PU_h = 0$ if and only if

$$a_j^\pm = c_j^\pm = b_0^\pm = a_0^\pm = d_0^\pm = 0 \qquad \text{for } j = 1, \ldots, p. \quad (34)$$

It follows that $U_h = \tilde{U}_h$. Finally, we have proved the following.

Lemma 5.3. *Assume $F = (0, f, 0)^T$ and $f \in C_b^0(\mathbb{R})$. Then the affine linear system (17) has a unique bounded solution $U_h \in C_b^0(\mathbb{D}_h) \bigcap C_b^1(\mathbb{H}_h)$ and the operator $K_h : C_b^0(\mathbb{R}) \to C_b^0(\mathbb{D}_h)$, $f \mapsto U_h$ is bounded.*

5.3. Affine equation in exponentially weighted spaces

The problem now is to extend lemma 5.3 to the case $f \in E_0^\alpha(\mathbb{R})$, with $\alpha > 0$ sufficiently close to 0. This has been done in [IK00] by constructing a suitable distribution space, but the following lemma gives an alternative proof (see [Mie87]).

Lemma 5.4. *Consider Banach spaces $\mathbb{D}, \mathbb{Y}$ and $\mathbb{X}$ such that: $\mathbb{D} \hookrightarrow \mathbb{Y} \hookrightarrow \mathbb{X}$. Let L be a closed linear operator in $\mathbb{X}$, of domain $\mathbb{D}$, such that the equation*

$$\frac{\mathrm{d}U}{\mathrm{d}t} = LU + f \quad (35)$$

admits for any fixed $f \in C_b^0(\mathbb{Y})$ a unique solution

$$U = Kf$$

in $C_b^0(\mathbb{D}) \bigcap C_b^1(\mathbb{X})$, with in addition $K \in \mathcal{L}(C_b^0(\mathbb{Y}), C_b^0(\mathbb{D}))$. Then there exists $\alpha_0 > 0$ such that if $0 \leqslant \alpha < \alpha_0$, for all $f \in E_0^\alpha(\mathbb{Y})$ the system (35) admits a unique solution in $E_0^\alpha(\mathbb{D}) \bigcap E_1^\alpha(\mathbb{X})$ with

$$\|U\|_{E_0^\alpha(\mathbb{D})} \leqslant C(\alpha) \|f\|_{E_0^\alpha(\mathbb{Y})}. \quad (36)$$

Applying this result to our problem yields the following result.

Proposition 5.5. *There exists $\alpha_0 > 0$ such that for all $F = (0, f, 0)^T$ with $f \in E_0^\alpha(\mathbb{R})$ and $\alpha \in [0, \alpha_0]$, the affine linear system (17) has a unique solution $U_h \in E_0^\alpha(\mathbb{D}_h) \bigcap E_1^\alpha(\mathbb{H}_h)$. Moreover, the operator K_h: $E_0^\alpha(\mathbb{R}) \to E_0^\alpha(\mathbb{D}_h)$, $f \mapsto U_h$ is bounded (uniformly in $\alpha \in [0, \alpha_0]$).*

Remark 5.6. It has to be remarked that the previous argument is also valid if the system involves more general linear coupling terms (to the third or farther neighbours). Indeed, the perturbation in the dispersion relation still involves $O(1)$ terms in k and as a consequence $N(\mathrm{i}k, A, T, \gamma)$ is still $O(k^2)$. Hence the regularity argument with the kernel G still holds. However, for the case of pulsating travelling waves (i.e. solutions of (1) satisfying $x_n(t) = x_{n-p}(t - T)$ for some $p > 1$), as was pointed out in [Sir05], one has to be more careful and one can use for instance the alternative argument developed in [Sir05] to overcome the fact that the dispersion relation does not have the suitable behaviour.

We now state the reduction theorems. As previously described, we will focus on the simplest bifurcations. We consider the three following different sets:

- Let Δ_0 be the set of parameters $(A, T, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ such that the centre spectrum of (6) consists of p pairs of simple eigenvalues $\pm \mathrm{i}q_k$ $(k = 1, ..., p)$ with $p \geqslant 1$ and a pair of double non-semisimple eigenvalues $\pm q_0$. We denote by $\bar{P}$ the projector on the associated invariant subspace.

- Let Δ_1 be the set of parameters $(A, T, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ such that the centre spectrum of (6) consists of only one pair of simple eigenvalues $\pm iq^*$. We denote by P_* the projector on the associated invariant subspace.
- Let Δ_2 be the set of parameters $(A, T, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ such that the centre spectrum of (6) consists of p pairs of simple eigenvalues $\pm iq_k$ for $k = 1, \dots, p$ with $p > 1$. We denote by $\hat{P}$ the projector on the associated invariant subspace.

Section 4 shows that these sets are not empty. The results of this section show that the assumptions of theorem 8.1 in the appendix are met. We can then state the following theorem, where Δ stands for either Δ_0, Δ_1 or Δ_2 and P for either $\bar{P}$, P_* or $\hat{P}$.

Theorem 5.1. *Fix $(A_0, T_0, \gamma_0) \in \Delta$ and $k \geqslant 1$. There exists a neighbourhood $\mathcal{U} \times \mathcal{V}$ of $(0, A_0, T_0, \gamma_0)$ in $\mathbb{D} \times \mathbb{R}^3$ and a map $\psi \in C_b^k(\mathbb{D}_c \times \mathbb{R}^3, \mathbb{D}_h)$ such that the following properties hold for all $(A, T, \gamma) \in \mathcal{V}$ (with $\psi(0, A, T, \gamma) = 0$, $D\psi(0, A_0, T_0, \gamma_0) = 0$).*

- *If $U : \mathbb{R} \to \mathbb{D}$ solves (5) and $U(t) \in \mathcal{U}$ $\forall t \in \mathbb{R}$ then $U_h(t) = \psi(U_c(t), A, T, \gamma)$ for all $t \in \mathbb{R}$ and U_c is a solution of*
$$\frac{\mathrm{d}U_c}{\mathrm{d}t} = \mathcal{L}_{A_0,T_0,\gamma_0} U_c + P\mathcal{N}_T(U_c + \psi(U_c, A, T, \gamma)). \tag{37}$$
- *If $U_c : \mathbb{R} \to \mathbb{D}_c$ is a solution of (37) with $U_c \in \mathcal{U}_c = P\mathcal{U}$ $\forall t \in \mathbb{R}$, then $U = U_c + \psi(U_c, A, T, \gamma)$ is a solution of (5).*
- *The map $\psi(\cdot, A, T, \gamma)$ commutes with R. Moreover, the reduced system (37) is reversible under R.*

6. Normal form computations

Equation (37) is still complicated and this section is devoted to the computation of a normal form for (37). Normal form theory (see for instance [IA98]) ensures that there exists a change of variables $U_c = \tilde{U}_c + \tilde{P}_{\gamma,T}(\tilde{U}_c)$ (here $U_c = PU$ and $\tilde{P}_{\gamma,T}$ is close to the identity). This simplifies the reduced equation on the centre manifold. In this section, we provide this normal form at order 3.

6.1. Normal forms

The linear operator $\mathcal{L}_{A,T,\gamma}$ restricted to the $(2p+4)$-dimensional subspace $\mathbb{D}_c$ has the following structure in the basis $(V_0, \hat{V}_0, V_1, \dots, V_p, \bar{V}_0, \bar{\hat{V}}_0, \bar{V}_1, \dots, \bar{V}_p)$ of eigenvectors in lemma 4.6

$$\mathcal{L}^c_{A,T,\gamma} = \begin{pmatrix} J_1 & & & \\ & \Lambda & & \\ & & J_2 & \\ & & & \Lambda' \end{pmatrix},$$

where

$$J_1 = \begin{pmatrix} iq_0 & 1 \\ 0 & iq_0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} -iq_0 & 1 \\ 0 & -iq_0 \end{pmatrix},$$
$$\Lambda = \begin{pmatrix} iq_1 & & \\ & \ddots & \\ & & iq_p \end{pmatrix}, \qquad \Lambda' = \begin{pmatrix} -iq_1 & & \\ & \ddots & \\ & & -iq_p \end{pmatrix}. \tag{38}$$

To compute the normal form, we exclude points of Δ_0 which are close to points where $sq_0 + \sum_{j=1}^{p} r_j q_j = 0$ for $s, r_j \in \mathbb{Z}$ and $0 < |s| + \sum_{j=1}^{p} |r_j| \leqslant 4$ which correspond to strong resonances. We still denote this new set by Δ_0. The normal form computation is very similar to [IK00], to which we refer for details.

In what follows we set $\tilde{U}_c = \mathcal{A}V_0 + \mathcal{B}\hat{V}_0 + \sum_{j=1}^{p} \mathcal{C}_j V_j + \bar{\mathcal{A}}\bar{V}_0 + \bar{\mathcal{B}}\hat{\bar{V}}_0 + \sum_{j=1}^{p} \bar{\mathcal{C}}_j \bar{V}_j$. The normal form of the evolution equation (5) restricted to the centre space up to order 3 is given in the following lemma.

Lemma 6.1. *The normal form of the reduced equation (37) at order 3 reads*

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\xi} &= \mathrm{i}q_0\mathcal{A} + \mathcal{B} + \mathrm{i}\mathcal{A}\mathcal{P} + O\left(\left(|\mathcal{A}| + |\mathcal{B}| + \sum_{j=1}^{p} |\mathcal{C}_j|\right)^4\right), \\
\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}\xi} &= \mathrm{i}q_0\mathcal{B} + \mathrm{i}\mathcal{B}\mathcal{P} + \mathcal{A}\mathcal{S} + O\left(\left(|\mathcal{A}| + |\mathcal{B}| + \sum_{j=1}^{p} |\mathcal{C}_j|\right)^4\right), \\
\frac{\mathrm{d}\mathcal{C}_j}{\mathrm{d}\xi} &= \mathrm{i}q_j\mathcal{C}_j + \mathrm{i}\mathcal{C}_j\mathcal{Q}_j + O\left(\left(|\mathcal{A}| + |\mathcal{B}| + \sum_{j=1}^{p} |\mathcal{C}_j|\right)^4\right) \quad \text{for } j = 1, \ldots, p,
\end{aligned} \tag{39}$$

where $\mathcal{P}$, $\mathcal{S}$, $\mathcal{Q}_k$ are polynomials of the variables $u_0 = \mathcal{A}\bar{\mathcal{A}}$, $v_0 = \mathrm{i}(\mathcal{A}\bar{\mathcal{B}} - \bar{\mathcal{A}}\mathcal{B})$, $u_j = \mathcal{C}_j\bar{\mathcal{C}}_j$, with real coefficients depending smoothly on the parameters (T, A, γ) in the neighbourhood of Δ_0. Furthermore, we have

$$\begin{aligned}
\mathcal{P}(u_0, v_0, u_1, \ldots, u_p) &= p_0'(A, T, \gamma) + \sum_{j=0}^{p} p_j u_j + p_{p+1} v_0, \\
\mathcal{S}(u_0, v_0, u_1, \ldots, u_p) &= s_0'(A, T, \gamma) + \sum_{j=0}^{p} s_j u_j + s_{p+1} v_0, \\
\mathcal{Q}_k(u_0, v_0, u_1, \ldots, u_p) &= \tilde{q}_{0,k}'(A, T, \gamma) + \sum_{l=0}^{p} \tilde{q}_{l,k} u_l + \tilde{q}_{p+1,k} v_0,
\end{aligned} \tag{40}$$

where p_0', s_0', $\tilde{q}_{0j}'$ vanish on Δ_0.

The truncated normal form (obtained by neglecting terms of orders 4 and higher) is integrable with the following first integrals

$$\begin{aligned}
&\mathcal{A}\bar{\mathcal{B}} - \bar{\mathcal{A}}\mathcal{B}, \\
&|\mathcal{B}|^2 - \int_0^{|\mathcal{A}|^2} \mathcal{S}(x, |\mathcal{C}_1|^2, \ldots, |\mathcal{C}_p|^2, \mathrm{i}(\mathcal{A}\bar{\mathcal{B}} - \bar{\mathcal{A}}\mathcal{B}))\, \mathrm{d}x, \\
&|\mathcal{C}_j|^2, \text{for } j = 1, \ldots, p.
\end{aligned} \tag{41}$$

The existence of homoclinic orbits is linked to the sign on the coefficient s_0 in the polynomial $\mathcal{S}$. We consider values of the parameters A and T so that $s_0'(A, T) > 0$. The following section is devoted to the computation of s_0.

6.2. Computation of the coefficient s_0

First we expand the evolution equation (5) as follows:

$$\begin{aligned}
\frac{\mathrm{d}U}{\mathrm{d}\xi} = {} & \mathcal{L}^{(0)}U + (A - A_0)\mathcal{L}^{(1)}U + (T - T_0)\mathcal{L}^{(2)}U + (\gamma - \gamma_0)\mathcal{L}^{(3)}U \\
& + M_2(U, U) + M_3(U, U, U) + \cdots,
\end{aligned} \tag{42}$$

where $\mathcal{L}^{(0)}$ is the linear operator $\mathcal{L}_{A,T,\gamma}$ for $(A_0, T_0, \gamma_0) \in \Delta_0$,

$$\mathcal{L}^{(1)} = \gamma T^2 \begin{pmatrix} 0 \\ -2\varphi + \Phi(2) + \Phi(-2) \\ 0 \end{pmatrix},$$

$$\mathcal{L}^{(2)} = 2T \begin{pmatrix} 0 \\ (-1 - 2\gamma(1 + A))\varphi + \gamma(\Phi(1) + \Phi(-1)) + \gamma A(\Phi(2) + \Phi(-2)) \\ 0, \end{pmatrix},$$

$$\mathcal{L}^{(3)} = T^2 \begin{pmatrix} 0 \\ -2(1 + A)\varphi + (\Phi(1) + \Phi(-1)) + A(\Phi(2) + \Phi(-2)) \\ 0, \end{pmatrix},$$

$$M_2(U, U) = aT^2 \begin{pmatrix} 0 \\ \varphi^2 \\ 0 \end{pmatrix}.$$

and

$$M_3(U, U, U) = bT^2 \begin{pmatrix} 0 \\ \varphi^3 \\ 0 \end{pmatrix}.$$

We now expand the solution U around (A_0, T_0, γ_0) in the following way

$$U = \mathcal{A}V_0 + \mathcal{B}\hat{V}_0 + \sum_{j=1}^{p} \mathcal{C}_j V_j + \bar{\mathcal{A}}\bar{V}_0 + \bar{\mathcal{B}}\bar{\hat{V}}_0 + \sum_{j=1}^{p} \bar{\mathcal{C}}_j \bar{V}_j + \sum (A - A_0)^m (T - T_0)^n (\gamma - \gamma_0)^l \mathcal{A}^{r_0} \mathcal{B}^{\hat{r_0}} \mathcal{C}_1^{r_1} \mathcal{C}_2^{r_2} \ldots \mathcal{C}_p^{r_p} \cdot \bar{\mathcal{A}}^{s_0} \bar{\mathcal{B}}^{\hat{s_0}} \bar{\mathcal{C}}_1^{s_1} \bar{\mathcal{C}}_2^{s_2} \ldots \bar{\mathcal{C}}_p^{s_p} \phi^{(m,n,l)}_{r_0\hat{r_0}r_1\ldots r_p s_0\hat{s_0}s_1\ldots s_p}. \tag{43}$$

Then we use the expression of the normal form and the expansion (42) to find the terms of order $\mathcal{A}^2, |\mathcal{A}|^2, \mathcal{A}|\mathcal{A}|^2$ by identification. We get the expressions (we omit the index $(m, n, l) = (0, 0, 0)$ in the notations)

$$\begin{aligned} (2\mathrm{i}q_0 - \mathcal{L}_{A_0,T_0,\gamma_0})\phi_{20\ldots000\ldots0} &= M_2(V_0, V_0), \\ -\mathcal{L}_{A_0,T_0,\gamma_0}\phi_{10\ldots010\ldots0} &= 2M_2(V_0, \bar{V}_0), \\ s_0\hat{V}_0 + \mathrm{i}p_0V_0 + (\mathrm{i}q_0 - \mathcal{L}_{A_0,T_0,\gamma_0})\phi_{20\ldots010\ldots0} &= 2M_2(\bar{V}_0, \phi_{20\ldots000\ldots0}) + 2M_2(V_0, \phi_{10\ldots010\ldots0}) \\ &\quad + 3M_3(V_0, V_0, \bar{V}_0). \end{aligned} \tag{44}$$

This leads to

$$\begin{aligned} \phi_{20\ldots000\ldots0} &= K_1 \begin{pmatrix} 1 \\ 2\mathrm{i}q_0 \\ \mathrm{e}^{2\mathrm{i}q_0v} \end{pmatrix}, \\ \phi_{10\ldots010\ldots0} &= 2a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \\ \phi_{20\ldots010\ldots0} &= \varphi \begin{pmatrix} 1 \\ \mathrm{i}q_0 \\ \mathrm{e}^{\mathrm{i}q_0v} \end{pmatrix} + \mathrm{i}p_0 \begin{pmatrix} 0 \\ 1 \\ v\mathrm{e}^{\mathrm{i}q_0v} \end{pmatrix} + \frac{s_0}{2} \begin{pmatrix} 0 \\ 0 \\ v^2\mathrm{e}^{\mathrm{i}q_0v} \end{pmatrix} \end{aligned} \tag{45}$$

with p_0 and φ still unknown and

$$K_1 = \frac{aT_0^2}{T_0^2(1+2\gamma(1+A_0)) - 4q_0^2 - 2T_0^2\gamma\cos(2q_0) - 2T_0^2\gamma A_0\cos(4q_0)}, \tag{46}$$

$$[1 - \gamma_0 T_0^2\cos(q_0) - 4A_0\gamma_0 T_0^2\cos(2q_0)]s_0 = T_0^2(2aK_1 + 4a^2 + 3b). \tag{47}$$

The other coefficients can be computed in a similar way.

7. Small amplitude solutions of the evolution equation (5)

This section is devoted to the study of the small amplitude solutions of the reduced equation (37). This relies mainly on the study of the normal form given below (see lemma 6.1).

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\xi} &= \mathrm{i}q_0\mathcal{A} + \mathcal{B} + \mathrm{i}\mathcal{A}\mathcal{P} + O\left(\left(|\mathcal{A}| + |\mathcal{B}| + \sum_{j=1}^{p}|\mathcal{C}_j|\right)^4\right),\\
\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}\xi} &= \mathrm{i}q_0\mathcal{B} + \mathrm{i}\mathcal{B}\mathcal{P} + \mathcal{A}\mathcal{S} + O\left(\left(|\mathcal{A}| + |\mathcal{B}| + \sum_{j=1}^{p}|\mathcal{C}_j|\right)^4\right),\\
\frac{\mathrm{d}\mathcal{C}_j}{\mathrm{d}\xi} &= \mathrm{i}q_j\mathcal{C}_j + \mathrm{i}\mathcal{C}_j\mathcal{Q}_j + O\left(\left(|\mathcal{A}| + |\mathcal{B}| + \sum_{j=1}^{p}|\mathcal{C}_j|\right)^4\right) \qquad \text{for } j = 1,\ldots,p.
\end{aligned} \tag{48}$$

Depending on the parameter regime, we study several types of solutions for equation (5): periodic, quasi-periodic and homoclinic solutions.

7.1. *Periodic solutions of the evolution equation (5)*

The normal form (48) then reduces in this case to

$$\frac{\mathrm{d}\mathcal{C}_1}{\mathrm{d}\xi} = \mathrm{i}q_*\mathcal{C}_1 + \mathrm{i}\mathcal{C}_1\mathcal{Q}_1 + O(|\mathcal{C}_1|^4)$$

together with the conjugate equation.

We now apply the centre Devaney–Lyapunov theorem to get the following result.

Theorem 7.1. *For every $(A, T, \gamma) \in \Delta_1$ and for U in a neighbourhood of the origin in $\mathbb{D}$, equation (5) reduces to a two-dimensional reversible smooth vector field and the set of solutions close to* 0 *of (5) constitutes a one-parameter family of periodic orbits.*

As a corollary of the previous theorem, one gets theorem 3.1.

7.2. *Quasi-periodic solutions of (5)*

We are now interested in quasi-periodic motions for equation (5). In this case, the normal form (48) writes

$$\frac{\mathrm{d}\mathcal{C}_j}{\mathrm{d}\xi} = \mathrm{i}q_j\mathcal{C}_j + \mathrm{i}\mathcal{C}_j\mathcal{Q}_j + O\left(\left(\sum_{k=1}^{p}|\mathcal{C}_k|\right)^4\right) \qquad \text{for } j = 1,\ldots,p \tag{49}$$

together with the conjugate equations. We first perform a change of unknowns for $j = 1,\ldots,p$

$$C_j(\xi) = \sqrt{\rho_j(\xi)}\mathrm{e}^{\mathrm{i}\theta_j(\xi)},$$

where ρ_j, θ_j are real-valued functions. Therefore, the normal form becomes

$$\frac{\mathrm{d}\rho_j}{\mathrm{d}\xi} = \text{h.o.t.},$$

$$\frac{\mathrm{d}\theta_j}{\mathrm{d}\xi} = q_j + \mathcal{Q}_j(\rho_1, \ldots, \rho_p) + \text{h.o.t.}.$$

Note here that the real-coefficient polynomials $\mathcal{Q}_j$ write

$$\mathcal{Q}_j(\rho_1, \ldots, \rho_p) = \tilde{q}'_{0,j}(A, T, \gamma) + \sum_{l=1}^{p} \tilde{q}_{l,j}\rho_l.$$

From the previous equation, several remarks have to be done:

- By standard centre manifold theory, it is a well-known fact that centre manifolds are not analytic and that usual standard KAM theorems cannot be applied. However, it is also a well-known fact that one can design KAM theorems for finitely differentiable systems (see, for instance, [GEdlL08] and references therein).
- The frequency of the torus is $\omega = (q_1, \ldots, q_p)$ and has to be taken Diophantine.
- The twist condition of the KAM theorem amounts to ensuring that

$$|\det \tilde{q}_{i,j}| \geqslant \kappa$$

for some $\kappa > 0$.

Considering the standard symplectic structure J in $\mathbb{R}^{2p}$, it is then easy to see that the previous system inherits a Hamiltonian structure $H = H_0 + \mathcal{R}$, where H_0 is integrable (in the Arnold–Liouville sense) and $\mathcal{R}$ stands for higher order terms.

We refer the reader to the book [dlL01] for an account on KAM theory. We can now state the theorem.

Theorem 7.2. *Assume that $(A, T, \gamma) \in \Delta_2$ and the vector $(q_1, \ldots, q_p)$ is Diophantine in $\mathbb{R}^p$. Assume furthermore that there exists a constant $\kappa > 0$ such that*

$$|det\, \tilde{q}_{i,j}| \geqslant \kappa.$$

Assume that U is in a sufficiently small neighbourhood of the origin in $\mathbb{D}$. Then equation (5) reduces to a $2p$-dimensional reversible smooth vector field and the set of solutions close to 0 of (5) constitutes a family of quasi-periodic orbits.

7.3. Homoclinic solutions of (5)

We now concentrate on homoclinic solutions (possibly to 0) for system (5). In this case, these solutions for the truncated normal form for $C_j = 0$ $(j = 1, \ldots, p)$ are given by, provided $s_0(A_0, T_0, \gamma_0) < 0$ and $s'_0(A, T, \gamma) > 0$

$$\mathcal{A}(\xi) = r_0(\xi)\mathrm{e}^{\mathrm{i}(q_0\xi+\psi(\xi)+\theta)}, \qquad \mathcal{B}(\xi) = r_1(\xi)\mathrm{e}^{\mathrm{i}(q_0\xi+\psi(\xi)+\theta)}, \tag{50}$$

where

$$\begin{aligned} r_0(\xi) &= \sqrt{\frac{2s_0}{-s'_0}}\operatorname{sech}(\xi\sqrt{s_0}), \\ r_1(\xi) &= \frac{\mathrm{d}r_0(\xi)}{\mathrm{d}\xi}, \\ \psi(\xi) &= p_0\xi - \frac{2p'_0\sqrt{s_0}}{s'_0}\tanh(\xi\sqrt{s'_0}). \end{aligned} \tag{51}$$

These orbits are reversible under R if one chooses θ equal to 0 or π. The question is now to study the persistence of the solution given by equations (50).

Let $(A_0, T_0, \gamma_0) \in \Delta_0$ where we have excluded from Δ_0 parameter values associated with strong resonances. Consider (A, T, γ) close to (A_0, T_0, γ_0). The linearized operator $\mathcal{L}_{A,T,\gamma}$ has four symmetric eigenvalues close to $\pm iq_0$, having non-zero real parts $(s_0'(A, T, \gamma) > 0)$ and several pairs of simple eigenvalues. We will first assume that (A, T, γ) is chosen in such a way that the linearized operator has only one pair of simple eigenvalues (see section 4). We also choose (A, T, γ) such that

$$s_0(A_0, T_0, \gamma_0) < 0.$$

Roughly speaking, (A, T, γ) is close to the *first* tongue in the parameter plane (see figure 1). We are now in a position to apply the results by Lombardi (see [Lom00, IL05]): homoclinic solutions to 0 do not persist for the full normal form (48). Instead, solutions homoclinic to exponentially small periodic orbits whose size is of the order $O(e^{-C/d^{1/2}})$, where d is the distance from (A, T, γ) to the bifurcation surface.

This allows to get the following theorem, which implies theorem 3.2.

Theorem 7.3. *Let (A, T, γ) be close to $(A_0, T_0, \gamma_0) \in \Delta_0$ (except for some exceptional points corresponding for instance to strong resonances) such that the centre spectrum consists of one pair of double non-semisimple eigenvalues and one pair of simple eigenvalues.*

Then the evolution equation (5) is reducible to a six dimensional vector field for U close to the origin in $\mathbb{D}$ and there are travelling waves with an exponentially small periodic tail provided that the coefficient depending only on constants of the problem satisfies $s_0(A_0, T_0, \gamma_0) < 0$.

In the previous situation where the centre spectrum consists of one pair of double non-semisimple and only one pair of simple eigenvalues, one can give a complete result thanks to Lombardi's results. However, in our case (see section 4), there are some regions in the parameter space for which the centre spectrum consists of one pair of double non-semisimple eigenvalue and *several* pairs of simple eigenvalues. In this context, a complete theory is not yet available. As in [JS05, Sir05], one can conjecture the persistence of homoclinic connections to higher-dimensional tori.

8. Formal multi-scale expansions

This section is devoted to a multi-scale expansion yielding a continuous approximation of the lattice equation (1). This type of analysis has been done by Remoissenet (from a formal point of view) in [Rem86] and justified in [GM04] in the case $A = 0$.

In this section, we use the approach of Remoissenet to get our continuous approximation for $A > 0$. However, it could be justified rigorously using the methods developed in [GM04]. The whole point consists of searching for a solution of (1) in the form of modulated plane waves. We write a solution of (1) as

$$x_n(t) = F_{1n}(t)e^{i\theta_n} + c.c. + \varepsilon\left[F_{0n}(t) + F_{2n}(t)e^{2i\theta_n} + c.c.\right] \tag{52}$$

with $\theta_n = q_0(n - t/T_0)$. We use a continuum approximation for the functions F_{jn} (replacing $F_{jn}(t)$ by $F_j(t, x)$). Making use of a multi-scale expansion as in [Rem86], we end up with the following nonlinear Schrödinger equation (NLS) given by

$$iF_{1s} + PF_{1ZZ} + Q|F_1|^2F_1 = 0, \tag{53}$$

where s, Z are slow variables (s stands for a time variable and Z for a spatial variable). The previous equation (53) is obtained by plugging (52) into (1) and identifying up to order ε^2 and the coefficients P and Q are given by the formulae

$$P = \frac{\gamma_0 T_0}{q_0}\left(\cos(q_0) + 4A\cos(2q_0) - \frac{\gamma_0 T_0^2}{q_0^2}(\sin(q_0) + 2A\sin(2q_0))^2\right) \tag{54}$$

and

$$Q = \frac{T_0}{2q_0}\left(4a^2 - \frac{2a^2}{3 + 16\gamma_0(\sin^4(q_0/2) + A\sin^4(q_0))} + 3b\right). \tag{55}$$

Now, using the dispersion relations (13) and (14), we note that

$$T_0^2(3 + 16\gamma_0(\sin^4(q_0/2) + A\sin^4(q_0))) = T_0^2(-1 - 2\gamma_0(1 + A_0)) + 4q_0^2 + 2\gamma_0 T_0^2\cos(2q_0) + 2A\gamma_0 T_0^2\cos(4q_0) \tag{56}$$

and

$$\gamma_0 T_0^2\left(\cos(q_0) + 4A_0\cos(2q_0) - \frac{\gamma_0 T_0^2}{q_0^2}(\sin(q_0) + 2A\sin(2q_0))^2\right) = (\gamma_0 T_0^2(\cos(q_0) + 4A_0\cos(q_0)) - 1). \tag{57}$$

So, we can express P and Q in the following way:

$$P = \frac{1}{q_0 T_0}\left(\gamma_0 T_0^2(\cos(q_0) + 4A_0\cos(q_0)) - 1\right) \tag{58}$$

and

$$Q = \frac{T_0}{2q_0}\left(4a^2 + 2aK_1 + 3b\right), \tag{59}$$

where K_1 is given by expression (46).

The existence of solitons in NLS for the specific wave number $q_0/2$ relies on the condition $PQ > 0$. Using expressions (58) and (59), one can see that this condition is equivalent to have $s_0 < 0$, where s_0 is given by (47).

It has to be noticed, and this is an important feature of the model, that the previous computations work for $A > 0$. The case $A < 0$ has also a physical meaning. It corresponds to anti-ferromagnetic interactions. In this case, an easy computation shows that the continuous approximation is no longer NLS but an elliptic equation involving a 4th derivative operator. We believe that the case $A < 0$ deserves more investigation and we postpone it to future work.

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Appendix: reduction to a centre manifold theorem

We recall here the theorem we use to reduce to a centre manifold. We state it as it is written in [VI92].

Let X, Y and Z be three Banach spaces with X continuously embedded in Y and Y continuously embedded in Z. Let $A \in \mathcal{L}(X, Z)$ and $g \in C^k(X, Y)$ for some $k \geqslant 1$. We consider the following differential equation

$$\dot{x} = Ax + g(x). \tag{60}$$

Before stating the theorem, we introduce some definitions and notations. Let E and F be two Banach spaces, $V \subset E$ an open subset, $k \in \mathbb{N}$ and $\eta \geqslant 0$. Then we define

$$C_b^k = \left\{ w \in C^k(V, F) \,|\, |w|_{j,V} = \sup_{x \in V} \|D^j w(x)\| < \infty,\ 0 \leqslant j \leqslant k \right\}$$

and

$$BC^\eta(\mathbb{R}, E) = \left\{ w \in C^0(\mathbb{R}, E) \,|\, \|w\|_\eta = \sup_{t \in \mathbb{R}} \mathrm{e}^{-\eta|t|} \|w(t)\|_E < \infty \right\}.$$

Theorem 8.1. *Assume that*

- *The function $g \in C^k(X, Y)$ for some $k \geqslant 1$ and $g(0) = 0$, $Dg(0) = 0$.*
- *There exists a continuous projection $\pi_c \in \mathcal{L}(Z, X)$ onto a finite-dimensional subspace $Z_c = X_c \subset X$ such that for all $x \in X$*

$$A\pi_c x = \pi_c A x$$

and such that if we set

$$Z_h = (I - \pi_c)Z, \quad X_h = (I - \pi_c)X, \quad Y_h = (I - \pi_c)Y,$$

$$A_c = A|_{X_c} \in \mathcal{L}(X_c), \quad A_h = A|_{X_h} \in \mathcal{L}(X_h, Z_h),$$

then the following holds:

(1) the spectrum of A_c is on the imaginary axis.

(2) There exists some $\beta > 0$ such that for each $\eta \in [0, \beta)$ and for each $f \in BC^\eta(\mathbb{R}, Y_h)$ the affine problem

$$\dot{x}_h = A_h x_h + f(t)$$

has a unique solution $x_h = K_h f$, where

$$K_h \in \mathcal{L}(BC^\eta(\mathbb{R}, Y_h), BC^\eta(\mathbb{R}, X_h))$$

for each $\eta \in [0, \beta)$ and

$$\|K_h\|_\eta \leqslant \gamma(\eta)$$

for all $\eta \in [0, \beta)$ and some continuous curve γ.

Then there exists a neighbourhood Ω of the origin in X and a mapping $\psi \in C_b^k(X_c, X_h)$ with $\psi(0) = 0$ and $D\psi(0) = 0$ such that

$$\mathcal{M} = \{x_c + \psi(x_c) \,|\, x_c \in X_c\}$$

is a local centre manifold for (60).

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