Quasicomponents in topos theory: the hyperpure, complete spread factorization

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Abstract

We establish the existence and uniqueness of a factorization for geometric morphisms that generalizes the pure, complete spread factorization for geometric morphisms with a locally connected domain. A complete spread with locally connected domain over a topos is a geometric counterpart of a Lawvere distribution on the topos, and the factorization itself is of the comprehensive type. The new factorization removes the topologically restrictive local connectedness requirement by working with quasicomponents in topos theory. In the special case when the codomain topos of the geometric morphism coincides with the base topos, the factorization gives the locale of quasicomponents of the domain topos, or its '0-dimensional' reflection.

Introduction

Our previous investigations into singular coverings of a topos [2, 3] are partly influenced by the complete spreads of R. H. Fox [7]. Such complete spreads are assumed to have a locally connected domain. We had proved that any geometric morphism with a locally connected domain admits a unique factorization into a pure geometric morphism followed by a complete spread with a locally connected domain, called the comprehensive factorization.

The main purpose of this paper is to show that the comprehensive factorization is an instance of a more general factorization in which it is not assumed that the domain topos is locally connected. In topology, E. Michael [10] has shown how to do this by following Fox's idea of considering cogerms of quasicomponents, and then by identifying the first factor in order to establish uniqueness.

In topos theory, quasicomponents are best viewed as the points of a locale of quasicomponents, i.e., as completely prime filters of definable subobjects. When the codomain topos is the base topos, the new factorization produces the locale of quasicomponents of the domain topos. It turns out that in this case completeness is not relevant: the structure geometric morphism of any topos is already complete. Intuitively, completeness is a 'horizontal' aspect of the topology of the map, as it is a property of how the map's fibers are glued together. On the other hand, the spread property is a 'vertical' aspect of the topology of the map, as it pertains to the topology of the fibers themselves. A map (or geometric morphism) into the base topos has only one fiber, so completeness is not pertinent.

The generalization of the comprehensive factorization to non-locally connected toposes not only sheds light on the locally connected case, but it also enlarges in principle the domain of applications of complete spreads. Several new questions naturally arise, not all of which are answered in this paper.

1 The comprehensive factorization revisited

Let $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ be a geometric morphism over $\mathscr{S}$, such that $\mathscr{F} \xrightarrow{f} \mathscr{S}$ is locally connected. Suppose that $\mathscr{E} = Sh(\mathbb{C}, J)$, and let

$$h : \mathbb{C} \longrightarrow P(\mathbb{C}) \longrightarrow \mathscr{E}$$

denote the Yoneda functor followed by sheafification. Throughout, $P(\mathbb{C})$ denotes the category of presheaves on $\mathbb{C}$. Let $\mathbb{Y}$ denote the category whose objects are pairs (C, α) such that $\alpha \rightarrowtail \psi^*(h_C)$ is a connected component. The functor $(C, \alpha) \mapsto C$ a discrete opfibration $\mathbb{Y} \longrightarrow \mathbb{C}$, which we may pullback to $\mathscr{E}$ producing *the comprehensive* *factorization* of ψ.

$$\begin{tikzcd}
\mathscr{F} \arrow[dr, "\rho"] \arrow[drr, "\zeta"] \arrow[ddr, "\psi"'] & & \\
& \mathscr{Y} \arrow[r, tail] \arrow[d, "\varphi"] \arrow[dr, phantom, "\lrcorner"] & P(\mathbb{Y}) \arrow[d] \\
& \mathscr{E} \arrow[r, tail] & P(\mathbb{C})
\end{tikzcd}$$

We have $\zeta^* h_{(C,\alpha)} = \alpha$, where $h : \mathbb{Y} \longrightarrow P(\mathbb{Y})$ denotes another Yoneda functor.

In order to construct this factorization in the absence of components we work with definable subobjects [1]. A *definable subobject* in $\mathscr{F}$ is a subobject $S \rightarrowtail T$ for which there is a pullback as follows.

$$\begin{tikzcd}
S \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & T \arrow[d] \\
f^*A \arrow[r, "f^*m"] & f^*B
\end{tikzcd}$$

But then m can be taken to be a monomorphism, and therefore classified by a map into $\Omega_{\mathscr{S}}$, so that a definable subobject may always be put in a pullback as follows.

$$\begin{tikzcd}
S \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & T \arrow[d] \\
f^*1 \arrow[r, tail, "f^*\top"] & f^*\Omega_{\mathscr{S}}
\end{tikzcd}$$

The map $T \longrightarrow f^*\Omega_{\mathscr{S}}$ may not be unique, but it is if $\mathscr{F}$ is *subopen*, meaning that the map $f^*\Omega_{\mathscr{S}} \longrightarrow \Omega_{\mathscr{F}}$ classifying $f^*\top$ is a monomorphism. Thus, in the subopen case, $f^*\top$ classifies definable subobjects.

Let $\mathbb{H}$ denote the category of pairs (C, U), such that $U \rightarrowtail \psi^*(h_C)$ is a definable subobject in $\mathscr{F}$. A morphism $(C, U) \xrightarrow{m} (D, V)$ of $\mathbb{H}$ is a morphism $h_C \xrightarrow{m} h_D$ in $\mathscr{E}$ such that the restriction of $\psi^*(m)$ to U factors through V. The category $\mathbb{Y}$ is a full subcategory of $\mathbb{H}$ since a component is a definable subobject. Let H denote $\psi_*(f^*\Omega_{\mathscr{S}})$ in $\mathscr{E}$. If

$\mathscr{F}$ is subopen, then a morphism $h_C \xrightarrow{U} H$ in $\mathscr{E}$ corresponds to such an object (C, U). We have a geometric morphism $\mathscr{F} \xrightarrow{\zeta} P(\mathbb{H})$ such that $\zeta^*(h_{(C,U)}) = U$, where again $h : \mathbb{H} \longrightarrow P(\mathbb{H})$ denotes the Yoneda functor.

The following observations provide the key to working without components.

Lemma 1.1 *Let $\mathbb{A}$ be a full subcategory of a category $\mathbb{B}$. Then a sieve $\{B_i \longrightarrow B\}$ is covering for the inclusion $P(\mathbb{A}) \rightarrowtail P(\mathbb{B})$ iff it contains all morphisms $A \longrightarrow B$, such that A is an object of $\mathbb{A}$.*

Proposition 1.2 *The Grothendieck topology in $\mathbb{H}$ that gives $P(\mathbb{Y})$ as a subtopos of $P(\mathbb{H})$ induced by the inclusion $\mathbb{Y} \rightarrowtail \mathbb{H}$ is generated by sieves*

$$\{(C, U_a) \xrightarrow{1_C} (C, U)\}_{a \in A}$$

such that $U = \bigvee_A U_a$ in $Sub_{\mathscr{F}}(\psi^*(h_C))$.

Proof. Apply Lemma 1.1 to $\mathbb{Y} \rightarrowtail \mathbb{H}$: $R = \{(C_a, U_a) \longrightarrow (C, U)\}$ covers iff R contains all morphisms $(D, \beta) \longrightarrow (C, U)$, where β is a component of $\psi^*(h_D)$. But this is true iff R contains all morphisms $(C, \alpha) \xrightarrow{1_C} (C, U)$. We use this equivalence to conclude the proof. A sieve $\{(C, U_a) \xrightarrow{1_C} (C, U)\}$ such that $U = \bigvee_A U_a$ certainly contains all morphisms $(C, \alpha) \xrightarrow{1_C} (C, U)$ because any component α must be contained in some definable U_a, so such a sieve is covering for the inclusion $P(\mathbb{Y}) \rightarrowtail P(\mathbb{H})$. On the other hand, suppose $R = \{(C_a, U_a) \longrightarrow (C, U)\}$ is covering for the inclusion, i.e., suppose R contains all morphisms $(C, \alpha) \xrightarrow{1_C} (C, U)$. Consider the sieve $S = \{(C, \alpha) \xrightarrow{1_C} (C, U)\}_{\alpha \in f_! U}$. We have $S \rightarrowtail R$, and certainly S has the required property $\bigvee_{f_! U} \alpha = U$. Therefore, R is a member of the topology generated by the sieves with the given property.

The sieves in Proposition 1.2 can be expressed with the following diagram in $\mathscr{F}$, in which the top horizontal morphism is an epimorphism, and the bottom square is pullback. Let V denote the coproduct

$\coprod_A U_a$ in $\mathscr{F}$.

$$\begin{tikzcd} V \arrow[r, two heads] \arrow[d, tail] & U \arrow[d, tail] \\ f^*A \times \psi^*(h_C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \psi^*(h_C) \arrow[d] \\ f^*A \arrow[r] & 1 \end{tikzcd} \tag{1}$$

Moreover, $V \rightarrowtail f^*A \times \psi^*(h_C)$ is a definable subobject.

Remark 1.3 The sieves (1) are included in the larger collection of sieves

$$\{(C_a, U_a) \xrightarrow{m_a} (C, U)\}_{a \in A}$$

that we may associate with the following diagram. Here E denotes the coproduct $\coprod_A h_{C_a}$ in $\mathscr{E}$, and $V = \coprod_A U_a$ in $\mathscr{F}$.

$$\begin{tikzcd} V \arrow[r, two heads] \arrow[d, tail] & U \arrow[d, tail] \\ \psi^*(E) \arrow[r, "\psi^* m"] \arrow[d] & \psi^*(h_C) \arrow[d] \\ f^*A \arrow[r] & 1 \end{tikzcd} \tag{2}$$

The larger collection (2) gives the image topos of $\mathscr{F} \xrightarrow{\zeta} P(\mathbb{H})$. The only and crucial difference between the sieves (1) and (2) is that in (1) the lower square is a pullback. The sieves (1) do not use connected components (this is the point of Proposition 1.2); however, this is a subtlety because in the locally connected case (1) may always be refined by a cover of connected components (the so-called

FC-coreflection [6]), as in the following diagram.

$$\begin{tikzcd}
V \arrow[r, equal] \arrow[d, tail] & V \arrow[r, two heads] \arrow[d, tail] & U \arrow[d, tail] \\
f^*f_!V \times \psi^*(h_C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & f^*A \times \psi^*(h_C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \psi^*(h_C) \arrow[d] \\
f^*f_!V \arrow[r] & f^*A \arrow[r] & 1
\end{tikzcd}$$

The outside square is again of the form (1).

We may regard the comprehensive factorization of ψ as two-step pullback.

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\rho"] \arrow[ddr, "\psi"'] & \mathscr{Y} \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner"] & P(\mathbb{Y}) \arrow[d, tail] \\
 & \mathscr{E}^{H^{\mathrm{op}}} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P(\mathbb{H}) \arrow[d] \\
 & \mathscr{E} \arrow[r, tail] & P(\mathbb{C})
\end{tikzcd}$$

In the above factorization, the middle topos $\mathscr{Y}$ is also locally connected, and the first factor ρ is what we call *pure*, meaning that the unit

$$e^*\Omega_{\mathscr{S}} \longrightarrow \rho_*\rho^*(e^*\Omega_{\mathscr{S}})$$

is an isomorphism. Purity can be thought as a bijection between the lattices of definable subobjects of E and of ρ^*E, for any object E of $\mathscr{E}$. The bijection is given in one direction by applying ρ^*, and in the other by the pullback in Lemma 1.4, 2, below. We invite the reader to prove this lemma.

Lemma 1.4 *If $\mathscr{F} \xrightarrow{\rho} \mathscr{E}$ is pure, then:*

1. *for any definable subobject $V \rightarrowtail E$ in $\mathscr{E}$, the naturality square*

$$\begin{tikzcd}
V \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner"] & \rho_*\rho^*V \arrow[d, tail] \\
E \arrow[r, "\eta"] & \rho_*\rho^*E
\end{tikzcd}$$

is a pullback (ρ is dense),

2. *for any definable subobject* $U \rightarrowtail \rho^* E$ *in* $\mathscr{F}$, *the pullback in* $\mathscr{E}$

$$\begin{tikzcd} V \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner"] & \rho_* U \arrow[d, tail] \\ E \arrow[r, "\eta"] & \rho_* \rho^* E \end{tikzcd}$$

is a definable suboject, and

3. *for any definable subobject* $U \rightarrowtail \rho^* E$ *in* $\mathscr{F}$, *the counit* $\rho^* \rho_* U \xrightarrow{\epsilon} U$ *has a section* $U \xrightarrow{s} \rho^* \rho_* U$ $(\epsilon \cdot s = 1_U)$ *such that the square*

$$\begin{tikzcd} U \arrow[r, tail, "s"] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner"] & \rho^* \rho_* U \arrow[d, tail] \\ \rho^* E \arrow[r, tail, "\rho^* \eta"] & \rho^* \rho_* \rho^* E \end{tikzcd}$$

is a pullback. (The section s is necessarily unique with this property.)

Conversely, if $\mathscr{F}$ *is subopen* ($f^* \Omega_{\mathscr{S}} \longrightarrow \Omega_{\mathscr{F}}$ is a monomorphism), *then the above three conditions imply that* ρ *is pure and that* $\mathscr{E}$ *is also subopen.*

Ultimately we shall characterize the first factor ρ in the general case (without local connectedness) by a property that comes from a topological condition first discovered by E. Michael. In general, the condition, which we call hyperpure, implies pure.

2 A new factorization

Let $\psi : \mathscr{F} \longrightarrow \mathscr{E}$ be a geometric morphism over $\mathscr{S}$. We formalize the covers (1) and (2). In so doing we incorporate the fibrational aspect [6] by dealing with families $U \rightarrowtail \psi^* C \xrightarrow{\psi^* y} f^* B$ of definable subobjects of $\psi^* C$, rather than with a single definable subobject $U \rightarrowtail \psi^*(h_C)$. Moreover, there is no need to consider sites in these definitions.

Definition 2.1 A *ψ-cover* ([6], Def. 6.1) is a diagram

$$\begin{tikzcd}
V \arrow[r, two heads] \arrow[d, tail] & U \arrow[d, tail] \\
\psi^*E \arrow[r, "\psi^*m"] \arrow[d, "\psi^*x"'] & \psi^*C \arrow[d, "\psi^*y"] \\
f^*A \arrow[r, "f^*l"] & f^*B
\end{tikzcd}$$

in $\mathscr{F}$, where $V \rightarrowtail \psi^*E$ and $U \rightarrowtail \psi^*C$ are definable subobjects. If the square

$$\begin{tikzcd}
E \arrow[r, "m"] \arrow[d, "x"'] & C \arrow[d, "y"] \\
e^*A \arrow[r, "e^*l"] & e^*B
\end{tikzcd}$$

coming from $\mathscr{E}$ is a pullback (so that m is definable), then we call the diagram a *weak ψ-cover.*

In the absence of local connectedness there is no category of components $\mathbb{Y}$. Nevertheless, there is a subtopos $\mathscr{Z} \rightarrowtail P(\mathbb{H})$ (generalizing $P(\mathbb{Y})$) of sheaves for the topology in $\mathbb{H}$ generated by the weak ψ-covers, whose description does not depend on connected components. The ψ-covers give the image topos of $\mathscr{F} \xrightarrow{\zeta} P(\mathbb{H})$, which must be contained in $\mathscr{Z}$. Then as before we may pullback to $\mathscr{E}$ producing the following diagram of geometric morphisms.

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\rho"] \arrow[ddr, "\psi"'] & \mathscr{X} \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner"] & \mathscr{Z} \arrow[d, tail, "\tau"] \\
& \mathscr{E}^{H^{\mathrm{op}}} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P(\mathbb{H}) \arrow[d] \\
& \mathscr{E} \arrow[r, tail] & P(\mathbb{C})
\end{tikzcd} \tag{3}$$

Thus, we have a factorization $\psi \cong \varphi \cdot \rho$ as above, where φ denotes the composite $\mathscr{X} \rightarrowtail \mathscr{E}^{H^{\mathrm{op}}} \longrightarrow \mathscr{E}$, which we shall call *the hyperpure, complete spread factorization of ψ*. We shall refer to the middle topos $\mathscr{X}$ as *the Michael topos of ψ.*

Proposition 2.2 *When the domain topos of a geometric morphism is locally connected, then its hyperpure, complete spread factorization coincides with the comprehensive factorization.*

Proof. This holds by construction, and by Proposition 1.2.

In what follows, by the hyperpure, complete spread factorization we shall always mean the factorization obtained by the canonical construction just described. The name for the new factorization suggests how its factors ρ and φ are identified. We shall do that later in order to formulate and establish the uniqueness of the factorization, but perhaps it is worthwhile to first examine the factorization in a special case, for the purposes of illustration.

3 The locale of quasicomponents

Let us analyse the hyperpure, complete spread factorization when $\mathscr{E}$ and $\mathscr{S}$ coincide, so that we wish to factor $\mathscr{F} \xrightarrow{f} \mathscr{S}$. If f is locally connected, then the factorization coincides with $\mathscr{F} \longrightarrow \mathscr{S}/f_!(1_{\mathscr{F}}) \longrightarrow \mathscr{S}$. In general, we obtain a factorization

$$\mathscr{F} \longrightarrow Sh(X) \longrightarrow \mathscr{S} \ ,$$

where X is what we call *the locale of quasicomponents of* $\mathscr{F}$.

In this case the object classifier $\Omega_{\mathscr{S}}$ serves as a site for $\mathscr{S}$, and $H = f_* f^* \Omega_{\mathscr{S}}$. We have the following topos pullbacks.

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\rho"] \arrow[ddr, "\psi"'] & Sh(X) \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner"] & \mathscr{Z} \arrow[d, tail] \\
& P(H) \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P(\mathbb{H}) \arrow[d] \\
& \mathscr{S} \arrow[r, tail] & P(\Omega_{\mathscr{S}})
\end{tikzcd}$$

For example, if $\mathscr{S} = Set$, then $\mathbb{H}$ is the lattice of all complemented subobjects $U \rightarrowtail 1_{\mathscr{F}}$ plus an extra object $(0,0)$ that is below the complemented subobject $0 \rightarrowtail 1_{\mathscr{F}}$. H is the same lattice as $\mathbb{H}$, except without the extra object $(0,0)$. The coverage in H giving the locale

X is simply the 'honest join' coverage: $\{U_a \rightarrowtail U\}$ covers just when $\bigvee U_a = U$ in $Sub_{\mathscr{F}}(1)$. (We call the join 'honest' because it is taken in $Sub_{\mathscr{F}}(1)$, not in H.) The frame $\mathscr{O}(X)$ can be identified as the lattice of subobjects $V \rightarrowtail 1_{\mathscr{F}}$ that are joins of complemented ones $U \rightarrowtail 1_{\mathscr{F}}$.

It follows that X is a 0-dimensional locale in the sense that

$$\Delta \dashv \Gamma : Sh(X) \longrightarrow \mathscr{S}$$

is a spread [2]. This means that there is a generating family $G \longrightarrow \Delta A$ that is a definable morphism. An equivalent condition is that in $Sh(X)$ the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner"] & 1 \arrow[d, "\Delta\top"] \\ \Delta\Gamma(\Delta\Omega_{\mathscr{S}}) \arrow[r] & \Delta\Omega_{\mathscr{S}} \end{tikzcd}$$

along the counit is a generating family. The locale of quasicomponents is a reflection (left adjoint) of locales into 0-dimensional locales.

The collection of open neighbourhoods of a point of a topological space is a filter (upclosed and closed under finite meets) that is inaccessible by joins, also said to be completely prime. Likewise, the collection of clopen subsets containing a quasicomponent of a point, which is the intersection of all clopen neighbourhoods of the point [11], is a filter that is inaccessible by honest joins of clopens. Such a collection is equivalently given as a point $1 \longrightarrow X$ of the locale of quasicomponents of $\mathscr{F}$. Thus, a point of the locale of quasicomponents appropriately generalizes the usual notion of quasicomponent. Moreover, the topos pullback

$$\begin{tikzcd} \mathscr{Q} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner"] & \mathscr{S} \arrow[d] \\ \mathscr{F} \arrow[r] & Sh(X) \end{tikzcd}$$

of such a point produces a 'quasicomponent subtopos' of $\mathscr{F}$. Of course, a quasicomponent subtopos may not be open, but as far as we know such a subtopos may not even be closed. (Quasicomponents of a topological space are always closed sets.) Conceivably, X may have no points at all.

4 Hyperpure geometric morphisms

In this section we shall identify the nature of the first factor of the new factorization under the name hyperpure. First consider the following example.

Example 4.1 The inclusion $f(x) = \sin(\frac{1}{x})$ of the positive reals into the topologist's sine curve

$$\{(0,y) \mid -1 \leq y \leq 1\} \cup \{(x, \sin(\frac{1}{x})) \mid x > 0\}$$

is a pure map in the sense explained at the end of § 1. But consider Michael's property: for every open U, every clopen covering $\{W_a\}$ of $f^{-1}U$ can be extended to a clopen covering $\{V_a\}$ of U such that $f^{-1}V_a = W_a$. The map f does not have this property. For instance, consider when U is a sufficiently small open disk that meets the line segment $-1 \leq y \leq 1$.

We next interpret Michael's property for geometric morphisms. Consider a diagram

$$\begin{tikzcd} W \arrow[r, two heads] \arrow[d, tail] & U \arrow[d, tail] \\ E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C \arrow[d] \\ e^*A \arrow[r] & e^*B \end{tikzcd} \tag{4}$$

in $\mathscr{E}$, where the lower square is a pullback, and the subobjects $W \rightarrowtail E$ and $U \rightarrowtail C$ are definable. We may apply ρ^* to (4) to obtain a weak ρ-cover in $\mathscr{F}$.

$$\begin{tikzcd} \rho^*W \arrow[r, two heads] \arrow[d, tail] & \rho^*U \arrow[d, tail] \\ \rho^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \rho^*C \arrow[d] \\ f^*A \arrow[r] & f^*B \end{tikzcd}$$

A *collective epimorphism* is a diagram

$$\begin{tikzcd} C' \arrow[r, two heads] \arrow[d] & C \arrow[d] \\ e^*B' \arrow[r, "e^*b"] & e^*B \end{tikzcd}$$

in which the top horizontal is an epimorphism. We say that one collective epimorphism refines another if the former factors through the latter.

Definition 4.2 We shall say that ρ is *hyperpure* if any weak ρ-cover

$$\begin{tikzcd} V \arrow[r, two heads] \arrow[d, tail] & \rho^*U \arrow[d, tail, "\rho^*u"] \\ \rho^*E \arrow[r, "\rho^*m"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \rho^*C \arrow[d] \\ f^*A \arrow[r] & f^*B \end{tikzcd} \tag{5}$$

in the sense of Definition 2.1 is given locally by a diagram (4) in $\mathscr{E}$. This means that there is a collective epimorphism

$$\begin{tikzcd} C' \arrow[r, two heads] \arrow[d] & C \arrow[d] \\ e^*B' \arrow[r] & e^*B \end{tikzcd}$$

in $\mathscr{E}$ such that the pullback of (5) to e^*B' is given by a diagram (4) over e^*B'.

We also require a uniqueness condition: for any two representations (4) of a given (5), the two witnessing collective epimorphisms have a common refining collective epimorphism such that the pullback of the two representing diagrams (4) to the refinement are equal.

Note: the local nature of Definition 4.2 is due to the existential quantifier in Michael's condition, and since existence in a topos is given only locally.

In the rest of this section we investigate the notion of hyperpure, and show that the first factor of the new factorization is hyperpure.

We assume throughout that the domain topos is *a definable dominance* in the sense that it is subopen and its definable subobjects compose, although these assumptions may not always be necessary. Any Grothendieck topos (bounded over *Set*) is a definable dominance.

We frequently appeal to the following 'easier' but equivalent formulation of hyperpure.

Lemma 4.3 *$\mathscr{F} \xrightarrow{\rho} \mathscr{E}$ is hyperpure iff every diagram*

$$\begin{tikzcd}
V \arrow[d, tail] \arrow[dr, two heads] & \\
\rho^* E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \rho^* C \arrow[d] \\
f^* A \arrow[r] & f^* B
\end{tikzcd}$$

in which $V \rightarrowtail \rho^* E$ is definable (which is a special weak ρ-cover) *is given locally by a diagram*

$$\begin{tikzcd}
W \arrow[d, tail] \arrow[dr, two heads] & \\
E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C \arrow[d] \\
e^* A \arrow[r] & e^* B
\end{tikzcd}$$

from $\mathscr{E}$, where $W \rightarrowtail E$ is definable, subject to the same uniqueness requirement.

Proof. This follows by some straightforward diagram chasing, assuming that definable subobjects compose in $\mathscr{F}$.

Proposition 4.4 *Hyperpure morphisms are closed under composition.*

Proof. A straightforward examination of the definitions reveals that this is so.

Proposition 4.5 *A connected geometric morphism is hyperpure.*

Proof. Suppose that $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ is connected. The direct image functor of a connected (in fact, of a pure) geometric morphism preserves definable subobjects. To any diagram

$$\begin{tikzcd} V \arrow[d, tail] \arrow[dr, two heads, "m"] & \\ \psi^* E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \psi^* C \arrow[d] \\ f^* A \arrow[r] & f^* B \end{tikzcd}$$

in which $V \rightarrowtail \psi^* E$ is definable, simply apply ψ_*, using the fact that every unit $X \longrightarrow \psi_* \psi^* X$ is an isomorphism. This gives a diagram

$$\begin{tikzcd} \psi_* V \arrow[d, tail] \arrow[dr, two heads, "\psi_* m"] & \\ E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C \arrow[d] \\ e^* A \arrow[r] & e^* B \end{tikzcd}$$

in which $\psi_* V \rightarrowtail E$ is definable, and moreover $\psi^* \psi_* V \cong V$ (Lemma 1.4). It then follows easily that $\psi_* m$ is an epimorphism.

Lemma 4.6 *If a definable subobject $X \rightarrowtail E$ factors through any other subobject $Y \rightarrowtail E$, then the factorization $X \rightarrowtail Y$ is definable. The graph of any morphism $X \longrightarrow e^* A$ is a definable subobject.*

Proof. The first statement is an easy exercise. The second statement holds because the graph of such a morphism is defined by the diagonal $A \rightarrowtail A \times A$.

Proposition 4.7 *The direct image functor of a hyperpure geometric morphism preserves $\mathscr{S}$-coproducts.*

Proof. Suppose that $\mathscr{F} \xrightarrow{\rho} \mathscr{E}$ is hyperpure. By the uniqueness clause in Definition 4.2, it follows that the unit $e^*\Omega_{\mathscr{S}} \longrightarrow \rho_*\rho^*(e^*\Omega_{\mathscr{S}})$ is a monomorphism. By a general fact [2], every unit $e^*A \longrightarrow \rho_*\rho^*(e^*A)$ must therefore be a monomorphism.

We next show that every such unit is an epimorphism. Let $X \xrightarrow{x} \rho_* f^* A$ be any generalized element, whose transpose we denote $\rho^* X \xrightarrow{\chi} f^* A$. The graph of χ is a definable subobject, depicted in the following diagram.

$$
\begin{tikzcd}
\rho^* X \arrow[d, tail] \arrow[dr, equal] \arrow[dd, bend right=60, "\chi"'] & \\
f^* A \times \rho^* X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \rho^* X \arrow[d] \\
f^* A \arrow[r] & 1_{\mathscr{F}}
\end{tikzcd}
$$

We apply the hyperpure hypothesis to this diagram: there is a collective epimorphism

$$
\begin{tikzcd}
Y \arrow[r, two heads, "m"] \arrow[d] & X \arrow[d] \\
e^* B \arrow[r] & 1_{\mathscr{E}}
\end{tikzcd}
$$

and a diagram

$$
\begin{tikzcd}
W \arrow[d, tail] \arrow[dr, two heads, "n"] \arrow[dd, bend right=60, "w"'] & \\
e^*(A \times B) \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d] \\
e^*(A \times B) \arrow[r] & e^* B
\end{tikzcd}
$$

in $\mathscr{E}$ (in which the subobject W is definable), such that the square below left is a pullback in $\mathscr{F}$.

$$
\begin{tikzcd}
\rho^* W \arrow[r, two heads, "\rho^*(mn)"] \arrow[d, "\rho^* w"'] \arrow[dr, phantom, "\lrcorner"] & \rho^* X \arrow[d, "\chi"] \\
f^*(A \times B) \arrow[r] & f^* A
\end{tikzcd}
\qquad
\begin{tikzcd}
\rho^* W \arrow[r, two heads, "\rho^*(mn)"] \arrow[dr, "\rho^* q"'] & \rho^* X \arrow[d, "\chi"] \\
 & f^* A
\end{tikzcd}
$$

(Note: $\rho^* n$ is an isomorphism as it is opposite $1_{\rho^* X}$ in a pullback along $\rho^* m$.) In particular, we may regard the square as the commuting triangle above right, where q denotes the composite $W \xrightarrow{w} e^*(A \times B) \longrightarrow e^* A$. This triangle transposes to the following commutative square in $\mathscr{E}$.

$$\begin{tikzcd} W \arrow[r, twoheadrightarrow, "mn"] \arrow[d, "q"'] & X \arrow[d, "x"] \\ e^* A \arrow[r] & \rho_* f^* A \end{tikzcd}$$

In this square the bottom horizontal is the unit in question. It is therefore an epimorphism, concluding the proof.

Remark 4.8 We have the implications

$$\text{connected} \Rightarrow \text{hyperpure} \Rightarrow \text{direct image preserves } \mathscr{S}\text{-coproducts}$$
$$\Rightarrow \text{pure} \Rightarrow \text{dense.}$$

All these notions except "connected" are sensitive to the base topos $\mathscr{S}$. For geometric morphisms to the base topos $\mathscr{S}$, the first three notions are equivalent. For geometric morphisms between locally connected toposes, the middle three notions are equivalent.

We return to the hyperpure, complete spread factorization of $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$. We have the usual category $\mathbb{H}$, its topos of presheaves $P(\mathbb{H}) \xrightarrow{\gamma} \mathscr{S}$, the subtopos $\tau : \mathscr{Z} \rightarrowtail P(\mathbb{H})$ for weak ψ-covers, and its structure morphism $\mathscr{Z} \xrightarrow{z} \mathscr{S}$, so that $z \cong \gamma \cdot \tau$.

Lemma 4.9 *For any morphism $(C, U) \xrightarrow{1_C} (C, V)$ of $\mathbb{H}$,*

$$\tau^* h_{(C,U)} \rightarrowtail \tau^* h_{(C,V)}$$

is a definable subobject in $\mathscr{Z}$.

Proof. By Lemma 4.6 (first statement), it suffices to show that for any object (C, U) of $\mathbb{H}$,

$$\tau^* h_{(C,U)} \rightarrowtail \tau^* h_{(C,\psi^* h_C)}$$

is a definable subobject in $\mathscr{X}$. Consider the diagram

$$\begin{tikzcd}
h_{(C,U)} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & R \arrow[r, tail] \arrow[d] & h_{(C,\psi^* h_C)} \\
1 \arrow[r, tail, "\gamma^*\top"] & \gamma^*(\Omega_{\mathscr{S}}) &
\end{tikzcd} \tag{6}$$

in $P(\mathbb{H})$, where $R = \coprod_{\omega\in\Omega_{\mathscr{S}}} h_{(C,U_\omega)}$ and

$$\begin{tikzcd}
U_\omega \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \psi^*(h_C) \arrow[d, "\chi"] \\
1_{\mathscr{F}} \arrow[r, tail, "f^*\omega"] & f^*\Omega_{\mathscr{S}}
\end{tikzcd}$$

is a pullback in $\mathscr{F}$ along the map χ classifying the definable subobject $U \rightarrowtail \psi^*(h_C)$. For instance, $U_\top = U$. The sieve R is fibered over $\gamma^*(\Omega_{\mathscr{S}})$, as in (6), making the square a pullback. Moreover, R is a weak ψ-cover. In fact, R corresponds to the diagram

$$\begin{tikzcd}
\psi^*(h_C) \arrow[d, tail] \arrow[dr, equal] \arrow[dd, bend right=60, "\chi"'] & \\
f^*\Omega_{\mathscr{S}} \times \psi^*(h_C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \psi^*(h_C) \arrow[d] \\
f^*\Omega_{\mathscr{S}} \arrow[r] & 1_{\mathscr{F}}
\end{tikzcd}$$

in $\mathscr{F}$. The subobject in this diagram is definable as it is the graph of χ. This concludes the proof because τ^* inverts $R \rightarrowtail h_{(C,\psi^* h_C)}$ and preserves the pullback in (6).

Lemma 4.10 *Suppose that a hyperpure geometric morphism $\mathscr{F} \longrightarrow \mathscr{E}$ factors through a subtopos $\mathscr{G} \rightarrowtail \mathscr{E}$. Then the factorization $\mathscr{F} \longrightarrow \mathscr{G}$ is hyperpure.*

Proof. Straightforward diagram chasing.

Proposition 4.11 *The first factor of the factorization constructed in* § 2 *is hyperpure.*

Proof. Let $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ be any geometric morphism such that $\mathscr{F} \xrightarrow{f} \mathscr{S}$ is subopen and definable subobjects compose in $\mathscr{F}$ (what we call a definable dominance). By Lemma 4.10, it suffices to show that ξ in

$$\begin{tikzcd} & \mathscr{F} \arrow[dl, "\rho"'] \arrow[d, "\xi"] \arrow[dr, "\zeta"] & \\ \mathscr{X} \arrow[r, tail] & \mathscr{Z} \arrow[r, tail, "\tau"'] & P(\mathbb{H}) \end{tikzcd}$$

is hyperpure (as in diagram (3)). We begin with an object $\tau^* h_{(C,U)}$ in $\mathscr{Z}$, and a diagram

$$\begin{tikzcd} V \arrow[d, tail] \arrow[dr] & \\ f^*A \times \xi^*(\tau^* h_{(C,U)}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \xi^*(\tau^* h_{(C,U)}) \arrow[d] \\ f^*A \arrow[r] & 1_{\mathscr{F}} \end{tikzcd}$$

in $\mathscr{F}$, where the subobject V is definable, and

$$\xi^*(\tau^* h_{(C,U)}) = \zeta^* h_{(C,U)} = U \ .$$

Thus,

$$V \cong \coprod_A V_a$$

for a family $\{V_a\}$ of definable subobjects of U. Assuming that definable subobjects compose in $\mathscr{F}$, each V_a is therefore a definable subobject of $\psi^*(h_C)$. This gives a weak ψ-cover $\{(C, V_a) \xrightarrow{1_C} (C,U)\}$ in $\mathbb{H}$. Let

$$W = \coprod_A \tau^* h_{(C,V_a)} \ .$$

By Lemma 4.9, for any $a \in A$,

$$\tau^* h_{(C,V_a)} \rightarrowtail \tau^* h_{(C,U)}$$

is a definable subobject in $\mathscr{Z}$. Then $W \rightarrowtail z^*A \times \tau^* h_{(C,U)}$ is definable, and moreover, the composite

$$W \rightarrowtail z^*A \times \tau^* h_{(C,U)} \longrightarrow \tau^* h_{(C,U)}$$

is an epimorphism in $\mathscr{X}$ because $\{(C, V_a) \xrightarrow{1_C} (C, U)\}$ is a weak ψ-cover. Obviously we have $\xi^* W \cong V$ so we are done in the case of an object $\tau^* h_{(C,U)}$.

We may easily extend the above argument first to a coproduct of a family of $\tau^* h_{(C,U)}$'s, and then to a quotient of such a coproduct keeping in mind the existential nature of Definition 4.2. This gives us the generality that Definition 4.2 requires.

5 Complete spreads and uniqueness

Proposition 5.1 *The hyperpure, complete spread factorization of a geometric morphism does not depend on the site chosen for the codomain topos.*

Proof. Let $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ be a geometric morphism. Let $\mathbb{C}$ and $\mathbb{C}'$ be two sites for a topos $\mathscr{E}$, with corresponding sites $\mathbb{H}$ and $\mathbb{H}'$ of definable subobjects. We may without loss of generality assume that $\mathbb{C}$ is a full subcategory of $\mathbb{C}'$ (so that $\mathbb{H}$ is a full subcategory of $\mathbb{H}'$), because any two sites may always be included in a third larger one. Then we have topos pullbacks

$$\begin{tikzcd} \mathscr{X} \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner"] & \mathscr{X}' \arrow[d, tail] \\ P(\mathbb{H}) \arrow[r, tail] & P(\mathbb{H}') \end{tikzcd}$$

where $\mathscr{X}$ and $\mathscr{X}'$ are subtoposes for the weak ψ-covers in $\mathbb{H}$ and $\mathbb{H}'$, respectively. Of course, both the pullback to $\mathscr{E}$ of $P(\mathbb{H})$ and of $P(\mathbb{H}')$ give $\mathscr{E}^{H^{\mathrm{op}}}$. This concludes the proof because the middle topos in the hyperpure, complete spread factorization of ψ is constructed as the pullback of any subtopos for the weak ψ-covers.

Definition 5.2 We shall say that a geometric morphism is *a complete spread* if its hyperpure factor (in the canonical construction) is an equivalence.

Proposition 5.3 *The second factor in the hyperpure, complete spread factorization is a complete spread* (in other words, the factorization is idempotent).

Proof. Let $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ be given, with factorization $\mathscr{F} \xrightarrow{\rho} \mathscr{X} \xrightarrow{\varphi} \mathscr{E}$. By Prop. 5.1, we may assume that the site for $\mathscr{E}$ we choose for the factorization of ψ is also the one we choose for the factorization of φ. The hyperpure ρ is pure, so it immediately follows that $\mathbb{H}_\psi$ and $\mathbb{H}_\varphi$ are equivalent categories. Moreover, under this equivalence, a weak ψ-cover corresponds to a weak φ-cover. Thus, the site $\mathbb{H}_\varphi$ with weak φ-covers must produce the same topos $\mathscr{X}$ again.

This completes the existence of the hyperpure, complete spread factorization. We will establish uniqueness by showing that any such given factorization must coincide with the one obtained by the canonical construction.

Lemma 5.4 *Suppose that $\mathscr{G} \xrightarrow{\rho} \mathscr{F}$ is hyperpure, and let $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ be any other geometric morphism with hyperpure, complete spread factorization*

$$\mathscr{F} \xrightarrow{\tau} \mathscr{X} \xrightarrow{\varphi} \mathscr{E} \; .$$

Then the hyperpure, complete spread factorization of $\psi \cdot \rho$ is

$$\mathscr{G} \xrightarrow{\tau \cdot \rho} \mathscr{X} \xrightarrow{\varphi} \mathscr{E} \; .$$

Proof. This comes down to showing (once a site for $\mathscr{E}$ is chosen) that the weak ψ-covers and the weak $(\psi \cdot \rho)$-covers generate the same topology in the definable subobject site $\mathbb{H}$. Hyperpure says that we may 'take off ρ^*,' and this is precisely what is needed in order to justify that the weak ψ-covers and the weak $(\psi \cdot \rho)$-covers generate the same topology.

The uniqueness of the hyperpure, complete spread factorization holds as follows. Let $\psi = \xi \cdot \zeta$, where ζ is hyperpure and ξ is a complete spread. By Lemma 5.4, the canonical construction of ψ must coincide with ζ followed by the canonical construction applied to ξ, which by definition of complete spread is just ξ again. This shows that the two factorizations of ψ coincide.

6 Concluding comments

The theory of complete spreads geometric morphisms is already well-developed in the case when the domain topos is locally connected [4].

Although the wider class of complete spreads that we have defined in this paper is largely unexplored, we believe it offers several possibilities for new research. We mention some aspects that may interest the reader.

We have not separately identified the completeness aspect of complete spreads in the general case. Presumably such an identification can be made in a manner that follows the locally connected case [6].

Let us say that a topos $\mathscr{E}$ is *locally quasiconnected* if it has enough quasicomponents in the precise sense that for every object F of $\mathscr{E}$, the locale of quasicomponents of $\mathscr{E}/F$ is spatial. A locally connected topos $f : \mathscr{F} \longrightarrow \mathscr{S}$ is locally quasiconnected because the locale of quasicomponents of $\mathscr{F}/F$ is the discrete locale $f_!(F)$, which is spatial. When the domain topos of $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ is locally quasiconnected, it may make sense in the constructions to consider a small category of quasicomponents $\mathbb{Q}$ replacing the category of components $\mathbb{Y}$. An object of $\mathbb{Q}$ is a pair (C, Q), where Q is a point of the locale of quasicomponents of $\mathscr{F}/\psi^* h_C$. There is no functor $\mathbb{Q} \longrightarrow \mathbb{H}$ (generalizing the inclusion $\mathbb{Y} \rightarrowtail \mathbb{H}$), but there is a geometric morphism $P(\mathbb{Q}) \xrightarrow{\tau} P(\mathbb{H})$ such that

$$\tau^* h_{(C,U)} = \coprod \{ h_{(C,Q)} \mid U \in Q \} \ .$$

This construction is always available, but if $\mathscr{F}$ is locally quasiconnected, then $\mathscr{F} \xrightarrow{\zeta} P(\mathbb{H})$ factors through the image of τ. We conjecture that a locally quasiconnected Grothendieck topos has the property that its locally connected coreflection [8] is non-trivial in the sense that its canonical morphism to the given topos is a surjection.

Pure geometric morphisms are the first factor of a different factorization, due to Johnstone [9] for $\mathscr{S} = \mathit{Set}$, in which the second factor is called an entire geometric morphism. The pure, entire factorization also holds over an arbitrary base topos $\mathscr{S}$ [5, 4] for geometric morphisms whose domain is a definable dominance. An entire geometric morphism is one that is defined by a Stone locale. Since hyperpure implies pure, it follows that an entire geometric morphism is a complete spread in the sense of Definition 5.2.

An object X of a definable dominance $\mathscr{E}$ may be completed by factoring the etalé morphism $\mathscr{E}/X \longrightarrow \mathscr{E}$ into its hyperpure and complete spread parts. Hitherto, this construction had been known only in the case of a locally connected topos $\mathscr{E}$.

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