MATH 141 - Calculus II

Summary Notes

bh2smith@gmail.com

These notes are intended to provide a quick, concise reference to the course material, but should not be considered as a sufficient replacement of the textbook and/or attendance in class.

Contents

1. Review

A few useful identities involving summation notation

(1)

$$\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \underbrace{(n+1)+(n+1)+\cdots+(n+1)}_{n/2 \text{ times}} = \frac{n(n+1)}{2}$$

(2)

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

(3)

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

We call i the *index of summation.*

(4) Let $f(x) = 2+3x^2$, then

$$\sum_{i=2}^{4} f(i) = \sum_{i=2}^{4}(2+3i^2) = 2+3(2)^2+2+3(3)^2+2+3(4)^2 = 93$$

(5) *Geometric series:*

$$\sum_{i=0}^{n-1} ar^i = a+ar+ar^2+\cdots+ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

(6) *Reindexing:*

$$\sum_{i=1}^{5} \frac{(-1)^{i+1}}{i^2} = \sum_{i=0}^{4} \frac{(-1)^{i+2}}{(i+1)^2}$$

(7) *Theorem:*

$$\sum_{i=1}^{n} (cf(x_i)+dg(x_i)) = c\sum_{i=1}^{n} f(x_i) + d\sum_{i=1}^{n} g(x_i)$$

for all constants c, d.

1.1. **Estimating the area of regions bounded by a curve $y = f(x)$ and the x-axis.** Let $f(x) = x^2$ on the interval $[1, 10]$. For the area under the curve, we subdivide the interval into n small intervals of size Δx and make rectangles of heights $f(x_i)$. Then an approximate area under the curve is given by the sum of the area of these approximating rectangles.

That is, $\Delta x = (10-1)/n = 9/n$ and the *sample points* are $x_i = 1 + i\Delta x = 1 + 9i/n$, so

$$\begin{aligned}
\text{Approximate area } &= \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} f(x_i)\cdot\Delta x \\
&= \sum_{i=1}^{n} (1+9i/n)^2 \cdot 9/n \\
&= \frac{9}{n}\sum_{i=1}^{n}\left(1 + \frac{18i}{n} + \frac{81i^2}{n^2}\right) \\
&= \frac{9}{n}\left(n + \frac{18}{n}\sum_{i=1}^{n} i + \frac{81}{n^2}\sum_{i=1}^{n} i^2\right) \\
&= \frac{9}{n}\left(n + \frac{18}{n}\frac{n(n+1)}{2} + \frac{81}{n^2}\frac{n(n+1)(2n+1)}{6}\right) \\
&= \frac{9}{n}\left(n + 9(n+1) + \frac{27(n+1)(2n+1)}{2n}\right) \\
&= 9 + 81 + \frac{81}{n} + \frac{243(2n^2+3n+1)}{2n^2} \\
&= 90 + \frac{81}{n} + 243 + \frac{243(3)}{2n} + \frac{243}{2n^2}
\end{aligned}$$

The actual area is given by allowing the number of rectangles to become infinite (hence, arbitrarily small), thus letting n tend to infinity. That is, the actual area is found as

$$\lim_{n\to\infty}\left(90 + \frac{81}{n} + 243 + \frac{243(3)}{2n} + \frac{243}{2n^2}\right) = 90 + 243 = 333.$$

Note: the choice of sample points were the right hand sides of the rectangles, which gave us an over-estimate. One could also have chosen the left-hand sides of the rectangles and obtained an under-estimate and both would have the same limit.

2. The Definite Integral

Let $f(x)$ be a continuous function on $[a,b]$ and let $\Delta x = \frac{b-a}{n}$ for a known integer $n > 0$. Let x_i^* be a sample point in each Δx interval , then the expression $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a *Riemann sum*over $[a,b]$. The number obtained by $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a *definite integral* of $f(x)$ over $[a,b]$. The definite integral is then denoted by

$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x} \tag{1}$$

where

- $f(x)$ is the *integrand*
- $\int$ is the *integral sign*
- a, b are the *limits of integration*

and this process is called integration. If $f(x)$ is a function on $[a,b]$ such that $\int_a^b f(x)dx$ exists, we say that $f(x)$ is *integrable* on $[a,b]$.

Theorem 1. *If $f(x)$ is continuous on $[a,b]$, then it is integrable on $[a,b]$ (i.e. $\int_a^b f(x)dx$ exists).*

Definition 1. *If $f(x)$ is continuous and non-negative on $[a,b]$, then we define the* area *of the region bounded by $f(x)$ and the x-axis over $[a,b]$ as being the positive number obtained by $\int_a^b f(x)dx$. In the case that $f(x)$ is negative, we take the absolute value and if $f(x) \geq 0$ on $[a,c]$ then $f(x) \leq 0$ on $[c,b]$ where $a < c < b$, then the area is $\int_a^c f(x)dx + \left(-\int_c^b f(x)dx\right)$.*

Example 1. *Using Riemann sums to find a formula for* $\int_a^b xdx$ *(where* $0 < a < b$*).*

$$\begin{aligned}
\int_a^b xdx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(a + \frac{(b-a)i}{n}\right) \cdot \frac{b-a}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(a\Delta x + i\Delta x^2\right) \\
&= \lim_{n\to\infty} \left(a\Delta x \sum_{i=1}^{n} 1 + \Delta x^2 \sum_{i=1}^{n} i\right) \\
&= \lim_{n\to\infty} \left(a\Delta x n + \Delta x^2 \frac{n(n+1)}{2}\right) \\
&= \lim_{n\to\infty} \left(a(b-a) + \frac{(b-a)^2}{n^2} \cdot \frac{n^2+n}{2}\right) \\
&= a(b-a) + \lim_{n\to\infty} \left(\frac{(b-a)^2}{2} + \frac{(b-a)^2}{2n}\right) \\
&= a(b-a) + \frac{(b-a)^2}{2} \\
&= \frac{b^2-a^2}{2}
\end{aligned}$$

2.1. **Properties of the definite integral.** We provide the following list of properties without proof. Note that, geometrically, these results should be intuitive.

(1)
$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

(2)
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

(3)
$$\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

(4)
$$\int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx$$

(5) if $f(x) \geq 0$ on $[a, b]$, then
$$\int_a^b f(x)dx \geq 0$$

(6) if $f(x)$ is *odd* (i.e. $f(-x) = -f(x)$), then
$$\int_{-a}^a f(x)dx = 0$$

(7) if $f(x)$ is *even* (i.e. $f(-x) = f(x)$), then
$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$$

(8) If $f(x) \geq g(x)$ on $[a, b]$ then
$$\int_a^b f(x)dx \geq \int_a^b g(x)dx,$$

(9) If $m \le f(x) \le M$ for all x on $[a,b]$, then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

(10)

$$\int_a^a f(x)dx = 0$$

Example 2. *Show that* $\int_2^3 \frac{1}{x^2+1}dx$ *is in the interval* $[1/10, 1/5]$.
Note, for $x \in [2,3]$ *that* $x^2+1 \in [5,10]$ *or equivalently that* $5 \le x^2+1 \le 10$ *for all* $x \in [2,3]$. *Hence, also,*

$$m := 1/10 \le \frac{1}{x^2+1} \le 1/5 =: M$$

and our properties guarantee, that

$$\frac{1}{10} = \frac{3-2}{10} \le \int_2^3 \frac{1}{x^2+1}dx \le \frac{1}{5}.$$

Example 3. *Evaluate;* $\int_{-1}^3 2|x|dx = 2\left[\int_{-1}^0 -xdx + \int_0^3 xdx\right] = 2(\frac{1}{2}+\frac{9}{2})$

3. Average Value of a Function

Suppose $f(x)$ is integrable on $[a,b]$ with $a < b$. The *average value* of f on $[a,b]$ is defined as

(2)
$$\boxed{A(f)_{[a,b]} = \frac{1}{b-a}\int_a^b f(x)dx}$$

Example 4. *Find the average value of* $f(x) = 100 - 3x^2$ *on* $[1,5]$.

$$A(f)_{[1,5]} = \frac{1}{5-1}\int_1^5 (100-3x^2)dx = \frac{1}{4}(276) = 69$$

Theorem 2 (Average-value Theorem)**.** *Suppose* f *is continuous on* $[a,b]$, *with* $a < b$. *Then there exists an* $x^* \in [a,b]$ *such that* $f(x^*) = A(f)_{[a,b]}$. *That is, a continuous function actually achieves its averages value somewhere within the interval* $[a,b]$.

Proof. Since $f(x)$ is continuous there exist a minimum and maximum at some $c, d \in [a,b]$ respectively . Then, certainly $f(c) \le f(x) \le f(d)$ for all $x \in [a,b]$. Now, from property 9, we have

$$f(c)(b-a) \le \int_a^b f(x)dx \le f(d)(b-a)$$

or equivalently (dividing by $(b-a)$)

$$f(c) \le \frac{1}{b-a}\int_a^b f(x)dx \le f(d).$$

Now, by the intermediate value theorem, there exists $x^* \in [c,d]$ such that $f(x^*) = \frac{1}{b-a}\int_a^b f(x)dx$. □

4. Fundamental Theorem of Calculus - Part I

Theorem 3 (FTC I)**.** *Suppose* $f(x)$ *is continuous on* $[a,b]$. *Let* $F(x)$ *be a function on* $[a,b]$ *defined as* $F(x) = \int_a^x f(t)dt$. *Then* $F'(x) = f(x)$.

Proof. Let $x \in [a,b]$ and proceed computing the derivative by first principles.

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x} f(t)dt + \int_x^{x+h} f(t)dt - \int_a^x f(t)dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt \end{aligned}$$

By the average value theorem there is an $x^* \in [x, x+h]$ so that $f(x^*) = \frac{1}{h}\int_x^{x+h} f(t)dt$ so then

$$F'(x) = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt = \lim_{h\to 0} f(x^*) = \lim_{x^*\to x} f(x^*) = f\left(\lim_{x^*\to x}\right) = f(x)$$

as required. □

Example 5. (1)

$$\frac{d}{dt}\int_0^t \sqrt{x^2+1}dt = \sqrt{t^2+1}$$

(2)

$$\frac{d}{dt}\int_t^{-3} \sin^2(x)dx = -\frac{d}{dt}\int_{-3}^{t} \sin^2(x)dx = -\sin^2(t)$$

(3)

$$\begin{aligned} \frac{d}{dt}\int_0^{t^2} \cos(x^2)dx &= \frac{d}{dt}\int_0^{u(t)} \cos(x^2)dx \\ &= \frac{d}{du}\int_0^{u} \cos(x^2)dx \cdot \frac{du}{dt} \\ &= \cos(u^2)\cdot 2t \\ &= 2t\cos(t^4) \end{aligned}$$

(4)

$$\frac{d}{dt}\int_{-2t}^{t} \frac{1}{1+x^2}dx = \frac{d}{dt}\left(-\int_0^{-2t} \frac{1}{1+x^2}dx + \int_0^t \frac{1}{1+x^2}dx\right) = -\frac{-2}{1+4t^2} + \frac{1}{1+t^2}$$

Lemma 1. *If f and g are two functions whose derivative is F, then $f(x) = g(x) + c$*

Proof. Given that $f'(x) = F(x) = g'(x)$, then

$$\frac{d}{dx}(f(x) - g(x)) = F(x) - F(x) = 0 \Rightarrow f(x) - g(x) = c$$

since the derivative of a constant is 0. □

5. Fundamental Theorem of Calculus II

Definition 2. *A function F is called an* antiderivative *of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.*

Theorem 4 (FTC II)**.** *Suppose $f(x)$ is continuous on $[a,b]$ and let $F(x)$ be* any *anti-derivative of $f(x)$ (i.e. $F'(x) = f(x)$). Then*

$$\boxed{\int_a^b f(x)dx = F(b) - F(a)} \tag{3}$$

Proof. Given that $f(x)$ is continuous on $[a,b]$ and $F'(x)=f(x)$, let $G(x)=\int_a^x f(t)dt$ so that $G'(x)=f(x)$ too. By the previous lemma, then, there is some constant c such that $G(x)=F(x)+c$. Now,

$$\int_a^b f(t)dt = G(b) = G(b)-0 = G(b)-G(a) = (F(b)+c)-(F(a)+c) = F(b)-F(a)$$

□

This theorem says that knowing any anti-derivative $f(x)$ allows us to compute any definite integral.

Example 6. *Now we are able to compute a few more definite integrals like*

$$\int_2^3 e^x dx = e^x|_2^3 = e^3 - e^2$$

and

$$\int_{-7}^{500} \sin x dx = -\cos(500) - (-\cos(-7))$$

Notation: If $F(x)$ is an antiderivative of $f(x)$ denote by $\int f(x)dx$ the family of antiderivatives $F(x)+C$ and call this the *indefinite integral* of $f(x)$.

5.1. **A few antiderivatives.**

- $\int kdx = kx + C$
- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$
- $\int e^x dx = e^x + C$
- $\int \frac{1}{x} dx = \ln|x| + C$
- $\int \sin x dx = -\cos x + C$
- $\int \cos x dx = \sin x + C$
- $\int \sec^2 dx = \tan x + C$
- $\int \sec x \tan x dx = \sec x + C$

Example 7. *Find the reigion bounded by* $\cos x$ *and the* x*-axis over the interval* $[0,\pi]$.
Note: $\int_0^\pi \cos x dx$ *won't work since the function is negative on the second half of this interval. Must use,*

$$A = A_1 + A_2 = \int_0^{\pi/2} \cos x dx + \left(-\int_{\pi/2}^{\pi} \cos x dx\right) = \sin x|_0^{\pi/2} - \sin x|_{\pi/2}^{\pi} = (1-0)-(0-1) = 2$$

Example 8. $\int_{-1}^1 \frac{1}{x^2}$ *does not exist since* $\frac{1}{x^2}$ *is not continuous at* 0 *and* (3) *does not apply.*

Definition 3. *Let* $y=f(x)$ *be continuous, we define the* differential *of* y*, denoted by* dy*, as being* $dy = f'(x)\Delta x$ *where* $\Delta x = x_1 - x$ *and* $x_1 > x$.

Observation: if $y=x$, then $dy = dx = \frac{dx}{dx}\cdot\Delta x = \Delta x$ so $\Delta x = dx$ and so, if $y=f(x)$ then $dy = f'(x)dx$.

6. Integration by Substitution

Theorem 5 (The substitution rule)**.** *Let* $u=g(x)$ *be differentiable and having as its image, the interval* I*. Suppose* $f(x)$ *is continuous on the interval* I*. Then,*

$$\boxed{\int f(g(x))\cdot g'(x)dx = \int f(u)du} \tag{4}$$

Proof. Given that $u=g(x)$, we are required to show that the general antiderivative of $f(g(x))\cdot g'(x)$ is the same as the general antiderivative (AD) of $f(u)$.

Suppose $F(u)$ is an AD of $f(u)$, then $\int f(u)du = F(u)+C = F(g(x))+C$.

Now we claim that $\int f(g(x))g'(x) = F(g(x))+C$. Indeed, it remains to show that $\frac{d}{dx}(F(g(x))+C = f(g(x))g'(x)$ which is true by the chain rule so the theorem holds. □

Remark 1. *What the substitution rule is saying is that*

$$\int \underbrace{f(g(x))}_{f(u)} \underbrace{g'(x)dx}_{du}$$

and thats it!

Example 9. *letting $u = x^3 + 9$ so that $du = 3x^2 dx$ which is equivalent to $x^2 dx = \frac{1}{3} du$, we find*

$$\begin{aligned} \int_0^3 x^2 \sqrt{x^3+9} dx &= \int_{x=0}^{x=3} \frac{1}{3} \sqrt{u} du \\ &= \frac{1}{3} \left(\frac{2}{3} u^{2/3} \right) |_{x=0}^{x=3} \\ &= \frac{2}{9} (x^3+9)^{2/3} |_0^3 = 42 \end{aligned}$$

Example 10. *Here will will use $u = \sin x$.*

$$\int \sin^3 x \cos x dx = \int u^3 = \frac{u^4}{4} + C = \frac{\sin^4 x}{4} + C$$

7. Integration by Parts

Consider two functions $u(x), v(x)$. The product rule says that

$$\frac{d}{dx}(u(x) \cdot v(x)) = u'(x)v(x) + u(x)v'(x).$$

Integrating both sides of this equation now says that

$$u(x)v(x) = \int [u(x)v(x)]' = \int u'(x)v(x)dx + \int u(x)v'(x)dx = \int v(x)du + \int u(x)dv.$$

Rearranging, that is

$$\int u(x)dv = u(x)v(x) - \int v(x)du$$

or in more compact notation

$$\boxed{\int udv = uv - \int vdu} \tag{5}$$

Recall that

$$\begin{aligned} \frac{d}{dx} \arcsin x &= \frac{1}{\sqrt{1-x^2}} \Rightarrow \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C \\ \frac{d}{dx} \arccos x &= \frac{-1}{\sqrt{1-x^2}} \Rightarrow -\int \frac{dx}{\sqrt{1-x^2}} = \arccos x + C \\ \frac{d}{dx} \arctan x &= \frac{1}{1+x^2} \Rightarrow \int \frac{dx}{1+x^2} = \arctan x + C \end{aligned}$$

Example 11. *Now for a long list of examples with solutions!*

(1)

$$\int \arcsin x dx = x \arcsin x + \sqrt{1-x^2} + C.$$

Solution: *With $u = \arcsin x, dv = dx$, so that $du = \frac{1}{\sqrt{1-x^2}}, v = x$ (using integration by parts - IBP) we find*

$$\int \arcsin x dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx.$$

Now, the substitution $y = 1 - x^2$ so that $dy = -2xdx$ or $xdx = -\frac{dy}{2}$ reveals the above as

$$x \arcsin x - \frac{1}{2} \int y^{-1/2} dy = x \arcsin x + \frac{1}{2} 2\sqrt{y} + C = x \arcsin x + \sqrt{1-x^2} + C.$$

(2)

$$\int x^2 e^{-x} dx = -e^{-x}(x^2 + 2x + 2) + C$$

Solution: *This will require two iterations of IBP. First we will have* $u = x^2$ *and* $dv = e^{-x}dx$ *so that* $du = 2xdx$ *and* $v = -e^{-x}$*. The first iteration becomes*

$$\int x^2 e^{-x} dx = -x^2 e^{-x} - \int (-2x) e^{-x} dx = -x^2 e^{-x} + 2 \int x e^{-x} dx.$$

For the second term we use $u = x, dv = e^{-x}dx$ *so that* $du = dx, v = -e^{-x}$ *and the above becomes*

$$-x^2 e^{-x} + 2\left[-xe^{-x} + \int e^{-x} dx\right] = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

as required.

(3)

$$\int e^{2x} \sin(3x) dx = \frac{13}{2} e^{2x} \left(2\sin(3x) - \frac{3}{2}\cos(3x)\right) + C$$

Solution: *This will also require two iterations of IBP, but has a slightly different solution.*

Let $u = \sin(3x), dv = e^{2x}dx$ *so* $du = 3\cos(3x)dx, v = \frac{1}{2}e^{2x}$*,*

$$\int e^{2x} \sin(3x) dx = \tfrac{1}{2} \sin(3x) e^{2x} - \tfrac{3}{2} \int e^{2x} \cos(3x) dx$$

Now for the second term, use $u = \cos(3x), dv = e^{2x}dx$ *so that* $du = -3\sin(3x)dx, v = \frac{1}{2}e^{2x}$ *and then the integral from the second term above becomes*

$$\int e^{2x} \cos(3x) dx = \tfrac{1}{2} \cos(3x) e^{2x} + \tfrac{3}{2} \int e^{2x} \sin(3x) dx.$$

Notice that the original integral $y = \int e^{2x} \sin(3x)dx$ *has reappeared. However, this doesn't mean we have gone in a circle since other terms have come out. Putting these together reads*

$$\begin{aligned} y &= \tfrac{1}{2} \sin(3x) e^{2x} - \tfrac{3}{2} \int e^{2x} \cos(3x) dx \\ &= \tfrac{1}{2} \sin(3x) e^{2x} - \tfrac{3}{2} \left[\tfrac{1}{2} \cos(3x) e^{2x} + \tfrac{3}{2} \int e^{2x} \sin(3x) dx\right] \\ &= \tfrac{e^{2x}}{2} \left(\sin(3x) - \tfrac{3}{2}\cos(3x)\right) - \tfrac{9}{4} y \end{aligned}$$

or in other words,

$$(1 + \tfrac{9}{4})y = \frac{e^{2x}}{2} \left(\sin(3x) - \tfrac{3}{2}\cos(3x)\right) \Rightarrow \boxed{y = \tfrac{2}{13} e^{2x} (\sin(3x) - \tfrac{3}{2}\cos(3x))}$$

(4)

$$\int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C$$

Solution: *Here again we will find a similar solution to the previous problem. Let* I *denote the value of interest,* $\int \sec^3 x dx$*, (for later when we need to rearrange). For IBP, use* $u = \sec x, dv = sec^2 x dx$ *so that* $du = \sec x \tan x dx, v = \tan x$*. Now*

$$\begin{aligned} I &= \sec x \tan x - \int \sec x \tan^2 x dx \\ &= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx \\ &= \sec x \tan x - \int \sec^3 x x dx + \int \sec x dx = \sec x \tan x - I + \ln|\sec x \tan x| + C \end{aligned}$$

So, rearranging,

$$2I = \sec x \tan x + \ln|\sec x \tan x| + C \Rightarrow \boxed{I = \frac{1}{2} (\sec x \tan x + \ln|\sec x \tan x|) + C}$$

8. Trigonometric Integrals

Recall and remember the following trigonometric identities:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x), \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x), \text{ and } \sin^2 x + \cos^2 x = 1$$

Now, we will

$$\boxed{\text{Evaluate: } \int \sin^m x \cos^n x dx}$$

by considering three different cases.

Solution:

(i) when n is odd;

Write $n = 2k + 1$, then by the substitution $u = \sin x$

$$\begin{aligned}\int \sin^m x \cos^{2k+1} x dx &= \int \sin^m x (\cos^2 x)^k \cos x dx \\ &= \int \sin^m x (1 - \sin^2 x)^k \cos x dx \\ &= \int u^m (1 - u^2)^k du\end{aligned}$$

and this is easy to evaluate (begin a polynomial).

(ii) when m is odd;

Write $m = 2k + 1$ and the substitution $u = \cos x$

$$\int (\sin^2 x)^k \cos^n x \sin x dx = -\int (1 - u^2)^k u^n du$$

(iii) when both m and n are even;

The identity

$$\cos^2 x = \tfrac{1}{2}(1 + \cos(2x)) \text{ or } \sin^2 x = \tfrac{1}{2}(1 - \cos(2x))$$

followed by

$$\sin x \sin y = \tfrac{1}{2}(\sin(x + y) + \sin(x - y)) \text{ or } \cos x \cos y = \tfrac{1}{2}(\cos(x + y) + \cos(x - y))$$

will bring one of the even powers down to either case (i) or case (ii).

and similarly,

$$\boxed{\text{Evaluate: } \int \tan^m x \sec^n x dx}$$

(i) If n is even, say $n = 2k$, then the substitution $u = \tan x$ and $du = \sec^2 x dx$ gives

$$\begin{aligned}\int \tan^m x (\sec^2 x)^k dx &= \int \tan^m \sec^{2k-2} x \sec^2 x dx \\ &= \int \tan^m (\sec^2 x)^{k-1} \sec^2 x dx \\ &= \int \tan^m (1 + \tan^2 x)^{k-1} \sec^2 x dx \\ &= \int u^m (1 + u^2)^{k-1} du\end{aligned}$$

(ii) If m is odd, then we will substitute $u = \sec x$ with $du = \sec x \tan x dx$ like so

$$\begin{aligned}\int \tan^{2k+1} x \sec^n x dx &= \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x dx \\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x dx \\ &= \int (u^2 - 1)^k u^{n-1} du\end{aligned}$$

In any other case, we won't find a closed expression for the integral $\int \tan^m x \sec^n x dx$.

Example 12. *Now for a list of examples with solutions!*

(1)

$$\begin{aligned}\int \sin^3 x \cos^2 x dx &= \int (1 - \cos^2 x) \cos^2 x \sin x dx, \qquad u = \cos x, du = -\sin x dx \\ &= -\int (1 - u^2) u^2 du \\ &= -\int (u^2 - u^4) du \\ &= \frac{u^5}{5} - \frac{u^3}{3} + C \\ &= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C\end{aligned}$$

(2)

$$\begin{aligned}\int \cos^5 x dx &= \int (1 - \sin^2 x)^2 \cos x dx, \qquad u = \sin x, du = \cos x dx \\ &= \int (1 - u^2)^2 du \\ &= \int (1 - 2u^2 + u^4) du \\ &- u - \frac{2u^3}{3} + \frac{u^5}{5} + C \\ &= \sin x - \frac{2 \sin^3 x}{3} + \frac{\sin^5 x}{5} + C\end{aligned}$$

(3)

$$\begin{aligned}\int \sin^2 x \cos^2 x dx &= \int (\sin x \cos x)^2 dx \\ &= \int \left(\tfrac{1}{2} \sin(2x)\right)^2 dx \\ &= \tfrac{1}{4} \int \sin^2(2x) dx \\ &= \tfrac{1}{4} \int \tfrac{1}{2}(1 - \cos(4x)) dx \\ &= \tfrac{1}{8} \int (1 - \cos(4x)) dx \\ &= \tfrac{1}{8} x + \tfrac{1}{8} \frac{\sin(4x)}{4} + C\end{aligned}$$

(4)

$$\begin{aligned}\int \tan^3 x \sec x dx &= \int (\sec^2 x - 1)\sec^2 x \sec x \tan x dx, \qquad u = \sec x, du = \sec x \tan x dx\\ &= \int (u^2-1)u^2 du\\ &= \frac{u^5}{5} - \frac{u^3}{3} + C\end{aligned}$$

(5)

$$\begin{aligned}\int \tan^4 x \sec^4 x dx &= \int \tan^4 x(\tan^2 x + 1)\sec^2 x dx, \qquad u = \tan x, du = \sec^2 x\\ &= \int u^4(u^2+1)du\\ &= \frac{u^7}{7} + \frac{u^5}{5} + C\end{aligned}$$

9. Trigonometric Substitution

This method will apply to integrands containing a^2+x^2, a^2-x^2 or x^2-a^2.
So, if

Integrand has	substitute	apply identity
a^2-x^2	$x = a\sin x, -\pi/2 \le x \le \pi/2$	$1-\sin^2\theta = \cos^2 x\theta$ or $\cos^2\theta = 1/2(1+\cos 2\theta)$
x^2-a^2	$x = a\sec x, -\pi/2 \le x \le \pi/2$ or $\pi \le x \le 3\pi/2$	$\sec^2\theta - 1 = \tan^2 x\theta$
a^2+x^2	$x = a\tan x, -\pi/2 \le x \le \pi/2$	$1+\tan^2\theta = \sec^2 x\theta$

Example 13. *These ideas will be best demonstrated through a list of examples*

(1) *Letting* $x=\sin\theta$ *so that on the interval* $-\pi/2 \le \theta \le \pi/2$ *we have* $\theta = \arcsin x$ *and* $dx = \cos\theta d\theta$, *then*

$$\begin{aligned}\int \frac{x^3}{\sqrt{1-x^2}}dx &= \int \frac{\sin^3\theta}{\sqrt{1-\sin^2\theta}}\cos\theta d\theta\\ &= \int \frac{\sin^3\theta}{|\cos\theta|}\cos\theta d\theta\\ &= \int \sin^3\theta d\theta\\ &= \int (1-\cos^2\theta)\sin\theta d\theta, \qquad \textit{let } u=\cos\theta, du = -\sin\theta d\theta\\ &= -\int (1-u^2)du\\ &= -u + \frac{u^3}{3} + C\\ & -\cos\theta + \frac{\cos^3\theta}{3} + C\\ &= -\sqrt{1-x^2} + \frac{\sqrt{1-x^2}^3}{3} + C\end{aligned}$$

(2) *The trigonometric substitution here is* $x = \frac{3}{2}\tan\theta$ *on* $-\pi/2 \le \theta \le \pi/2$ *so that* $\theta = \arctan(2x/3)$ *and* $dx = \frac{3}{2}\sec^2\theta d\theta$*, then*

$$\begin{aligned}
\int \frac{1}{(4x^2+9)^2}dx &= \int \frac{1}{(4(\frac{3}{2}\tan\theta)^2+9)^2}(\tfrac{3}{2}\sec^2\theta)d\theta \\
&= \tfrac{1}{81}\cdot\tfrac{3}{2}\int \frac{\sec^2\theta}{\sec^4\theta}d\theta \\
&= \tfrac{1}{54}\int \frac{1}{\sec^2\theta}d\theta \\
&= \tfrac{1}{54}\int \cos^2\theta d\theta \\
&= \tfrac{1}{54}\int \tfrac{1}{2}(1+\cos(2\theta))d\theta \\
&= \tfrac{1}{108}\int (1+\cos(2\theta))d\theta \\
&= \tfrac{1}{108}(\theta + \tfrac{1}{2}\sin(2\theta)) + C \\
&= \tfrac{1}{108}(\arctan x + \tfrac{1}{2}2\sin\theta\cos\theta) + C \\
&= \tfrac{1}{108}(\arctan x + \sin\theta\cos\theta) + C \\
&= \tfrac{1}{108}(\arctan x + \tfrac{2x}{4x^2+9}\cdot\tfrac{3}{4x^2+9}) + C
\end{aligned}$$

where the last line is found from the fact that $\tan\theta = \frac{2x}{3}$ *so the hypotenuse is necessarily* $4x^2+9$ *hence* $\cos\theta = \frac{3}{4x^2+9}$ *and* $\sin\theta = \frac{2x}{4x^2+9}$.

(3) *Here we use* $x = \sin\theta$ *on* $\pi/2 \le \theta \le \pi/2$ *so that* $dx = \cos\theta d\theta$*. Note that this is definite integral so we have to change the bounds of integration. That is, when* $x = 1/2, \theta = \pi/6$ *and when* $x = \sqrt{3}/2, \theta = \pi/3$.

$$\begin{aligned}
\int_{1/2}^{\sqrt{3}/2} \frac{x^3}{\sqrt{1-x^2}}dx &= \int_{\pi/6}^{\pi/3} \frac{\sin^3\theta\cos\theta}{\sqrt{1-\sin^2\theta}}d\theta \\
&= \int_{\pi/6}^{\pi/3} \sin^3\theta d\theta, \ \textit{let } u = \cos\theta \\
&= \int_{\pi/6}^{\pi/3} (1-\cos^2\theta)\sin\theta d\theta, \ \textit{let } u = \cos\theta \\
&= -\int_{u=\sqrt{3}/2}^{u=1/2} (1-u^2)du \\
&= \left[-u + \frac{u^3}{3}\right]_{\sqrt{3}/2}^{1/2} \\
&= \left(\tfrac{1}{24} - \tfrac{1}{2}\right) - \left(\tfrac{3\sqrt{3}}{24} - \tfrac{\sqrt{3}}{2}\right) \\
&= \tfrac{9\sqrt{3}-11}{24}
\end{aligned}$$

(4) *Here we will use* $x+1 = 2\sin\theta$ *for* $|\theta| \le \pi/2$ *so* $\theta = \arcsin\left(\frac{x+1}{2}\right)$ *and* $dx = 2\cos\theta d\theta$. *Now*

$$\begin{aligned}\int \frac{x}{\sqrt{3-2x-x^2}}dx &= \int \frac{x}{4-(x+1)^2}dx \\ &= \int \frac{(2\sin\theta - 1)2\cos\theta}{\sqrt{4(1-\sin^2\theta)}}d\theta \\ &= \int \frac{2\sin\theta - 1)\cos\theta}{\cos\theta}d\theta \\ &= 2\int \sin\theta d\theta - \int d\theta \\ &= -2\cos\theta - \theta + C \\ &= -2\sqrt{4-(x+1)^2} - \arcsin\left(\tfrac{x+1}{2}\right) + C\end{aligned}$$

10. Integration with Partial Fractions

This method is used to solve a rational integrand (i.e. of the form $\frac{p(x)}{q(x)}$ for polynomials p and q).
Case 1: $\deg p \ge \deg q$, use long division.
Case 2: $\deg p < \deg q$, factor the denominator and split up with partial fractions.

Example 14. *As usual, this is best demonstrated by example.*

(1)

$$\begin{aligned}\int \frac{x^3+x^2+x-1}{x^2+2x+2}dx &= \int (x-1)dx + \int \frac{x+1}{x^2+2x+2}dx, \qquad \textit{(long division)} \\ &= \frac{x^2}{2} - x + \frac{\ln|x^2+2x+2|}{2} + C\end{aligned}$$

(2)

$$\int \frac{4x^2-3x-4}{x^3+x^2-2x}dx = \int \frac{4x^2-3x-4}{x(x-1)(x-2)}dx$$

and here we write

$$\begin{aligned}\frac{4x^2-3x-4}{x(x-1)(x-2)} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2} \\ &= \frac{A(x-1)(x+2) + Bx(x+2) + Cx(x-1)}{x(x-1)(x+2)} \\ &= \frac{(A+B+C)x^2 + (A+2B-C)x - 2A}{x(x-1)(x+2)}.\end{aligned}$$

This is the linear system

$$A+B+C = 4,$$
$$A+2B-C = -3$$

and

$$-2A = -4$$

which has solution $A = 2, B = -1, C = 3$. *So,*

$$\begin{aligned}\int \frac{4x^2-3x-4}{x^3+x^2-2x}dx &= \int \frac{4x^2-3x-4}{x(x-1)(x-2)}dx \\ &= \int \frac{2}{x}dx - \int \frac{1}{x-1}dx + \int \frac{3}{x+2}dx \\ &= 2\ln|x| - ln|x-1| + 3ln|x+2|_C\end{aligned}$$

(3) *Using that*

$$\frac{x^3+4x-1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

has solution $A = 1, B = 0, C = 3, D = -4$, *we find that*

$$\begin{aligned}\int \frac{x^3+4x-1}{x(x-1)^3}dx &= \int \frac{1}{x}dx + 3\int \frac{1}{(x-1)^2}dx - 4\int \frac{1}{(x-1)^3}dx \\ &= \ln|x| - 3(x-1)^{-1} + 2(x-1)^{-2} + C\end{aligned}$$

where we have used the substitution $u = x-1, du = dx$ *to integrate the second two terms.*

(4) *Here, the partial fractions look slightly different because of the irreducible quadratic factor in the denominator*

$$\frac{5x^3-3x^2+2x-1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}.$$

Notice, the linear $Cx+D$ *above the quadratic denominator. This is solved the same way as above and we find that* $A = 2, B = -1, C = 3, D = -2$, *so*

$$\begin{aligned}\int \frac{5x^3-3x^2+2x-1}{x^2(x^2+1)}dx &= 2\int \frac{1}{x}dx - \int \frac{1}{x^2}dx + \int \frac{3x-2}{x^2+1}dx \\ &= 2\ln|x| + \frac{1}{x} + 3\int \frac{x}{x^2+1} - 2\int \frac{1}{x^2+1}dx \\ &= 2\ln|x| + \frac{1}{x} + \tfrac{3}{2}\ln|x^2+1| - 2\arctan x + C\end{aligned}$$

11. Non-Elementary Functions

The *elementary functions* are polynomials, trigonometric, exponential, rational or any combination of these using $+, -, \times, \div$, exp or composition.

Consider, the function $F : \mathbb{N} \to \mathbb{N}$ defined by the rule that $F(n) =$ the n^{th} Fibonacci number. Is F an elementary function? The answer here is YES! Now, is every function elementary? NO.

For example, $f(t) := e^{t^2}$ is continuous on $\mathbb{R}$, so integrable. Then define $F(x) := \int_0^x e^{t^2}dt$ which has derivative $F'(x) = f(x)$ (by FTC I). It can be shown that $F(x)$ is not elementary, meaning you can't find a formula for $F(x)$.

12. Areas between curves

Consider a region bounded above by $y = f(x)$ and below by $y = g(x)$.

Case 1:

$$\begin{aligned}A &= A_1 - A_2 \\ &= \int_a^b f(x)dx - \int_a^b g(x)dx \\ &= \int_a^b (f(x) - g(x))dx\end{aligned}$$

Case 2:

$$\begin{aligned}A &= A_1 + A_2 \\ &= \int_a^b f(x)dx + \left(-\int_a^b g(x)dx\right) \\ &= \int_a^b (f(x) - g(x))dx\end{aligned}$$

Case 3:

$$\begin{aligned} A &= A_1 - A_2 \\ &= -\int_a^b g(x)dx - \left(-\int_a^b f(x)dx\right) \\ &= \int_a^b (f(x) - g(x))dx \end{aligned}$$

Notice that in all three cases thus far, we have the area between f and g on $[a, b]$ when $f(x) \geq g(x)$ is given by the formula

$$\int_a^b (f(x) - g(x))dx.$$

Now, suppose that $f(x) \geq g(x)$ on $[a, c]$ and then $g(x) \geq f(x)$ on $[c, b]$. Then

Case 4:

$$\begin{aligned} A &= A_1 + A_2 \\ &= \int_a^c (f(x) - g(x))dx + \int_c^b (g(x) - f(x))dx \\ &= \int_a^b (f(x) - g(x))dx \end{aligned}$$

Example 15. *Find the area of the region bounded by...*

(1) $Y = x$ *and* $y = 6 - x^2$*;*
Solution: *Since there is no given interval on which to find the area, we must compute where these two curves meet. That is, solve* $x = y = 6 - x^2$ *which is equivalent to* $0 = (x+3)(x-2)$ *so that these curves meet at* $x = -3$ *and* $x = 2$*. On this interval we find that* $6 - x^2 \geq x$ *so the area between them is computed by*

$$\int_{-3}^2 (6 - x^2 - x)dx = \left[6x - \frac{x^3}{3} - \frac{x^2}{2}\right]|_{-3}^2 = \tfrac{125}{6}$$

(2) $y = \frac{x}{2}$ *and* $y^2 = 8 - x$*. These curves meet when* $\frac{x^2}{4} = 8 - x$ *which has solutions* $x = -8$ *and* $x = 4$
Solution 1: *Here we integrate along the* x*-axis as follows;*

$$A = \int_{-8}^4 \left(\frac{x}{2} - (-\sqrt{8-x})\right) dx + 2\int_4^8 \sqrt{8-x}dx$$

Solution 2: *If we tilt our heads, this is actually easier done by integrating along the* y*-axis. The intersection points are* $(-8, -4)$ *and* $(4, 2)$ *and the area is given by*

$$\int_{-4}^2 (8 - y^2 - 2y)dy = (8y - \tfrac{y^3}{3} - y^2)|_{-4}^2$$

13. Calculating Volumes

13.1. **Cross sectional method.** The cross sectional method is often the most practical method used to compute volumes of *solids of revolution* (a solid obtained by revolving a region bounded by $y = f(x)$ over $[a, b]$ about the x- or y-axis).

Principle behind the cross section method

(1) subdivide $[a, b]$ into n equal sub intervals, each of length $\Delta x = \frac{b-a}{n}$
(2) Let $A(x_i)$ denote the area of the cross section of the solid at x_i
(3) Then $A(x_1), A(x_2), \ldots$ is a sequence of areas of cross sections, each a function of x_i.
(4) If we then keep $A(x)$ constant over the interval (x_i, x_{i+1}) we obtain an approximation $V_n(x_i) = A(x_i)\Delta x$
(5) We now have a sequence of volumes $V_n(x_1), V_n(x_2), \ldots$

(6) The n^{th} approximation of the volume of this solid is then

$$V_n = \sum_{i=1}^{n} V_n(x_i) = \sum_{i=1}^{n} A(x_i)\Delta x$$

which is a Riemann sum.

(7) So, theoretically, the actual volume of the solid is

$$V = \int_a^b A(x_i)dx = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\Delta x$$

Example 16. *Using the cross sectional method to calculate the volume of a square based pyramid, with base $b \times b$ and height h.*

Solution: *The side length at x can be found using similar triangles as $s(x) = \frac{xb}{h}$, so then the area at x is then $A(x) = s(x)^2 = \frac{x^2b^2}{h^2}$. Now, the volume can be computed by*

$$V = \int_0^h A(x)dx = \frac{b^2}{h^2}\int_0^h x^2dx = \frac{b^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{b^2h}{3}$$

Note that this method would also work using the y-axis and integrating with respect to y.

The above example was not a solid of revolution. However, there is a nice formula for the cross sectional area (and hence the volume) of a solid of revolution.

Notice, that when $y = f(x)$ is revolved about, say, the x-axis, the cross sectional area is a circle, of radius $f(x)$ centred at the x axis. So then $A(x) = \pi f(x)^2$ (the area of a circle of radius $f(x)$). Hence,

$$V = \int_a^b \pi f(x)^2 dx \tag{6}$$

If this was about the y-axis then $V = \int_c^d \pi g(y)dy$.

Example 17. *All of the following are solids of revolution.*

(1) *Show that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.*
Solution: *Here we use the function $y = \sqrt{r^2 - x^2}$ which is the top half of the circle to be revolved. So then the volume is achieved by integrating $A(x) = \pi y^2$ along the interval $[-r, r]$. That is,*

$$V = \int_{-r}^{r} \pi(\sqrt{r^2 - x^2})^2dx = \pi\int_{-r}^{r}(r^2 - x^2)dx = \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r} = \frac{4}{3}\pi r^2$$

(2) *Find the volume of a cone with height h and radius r.*
Solution: *This will be done along the y-axis, since standing cones make more sense than sideways cones. The relationship between x and y here is that $y = \frac{h}{r}x$ (which can be thought of as the slope of a cone whose tip is at the origin). So the radius at y is $x = \frac{r}{h}y$ and the volume is then given by*

$$V = \pi\int_0^h \left(\frac{r}{h}y\right)^2 dy = \pi\frac{r^2}{h^2}\int_0^h y^2dy = \pi\frac{r^2}{h^2}\cdot\frac{h^3}{3} = \frac{1}{3}\pi r^2 h$$

Now, we are ready for something slightly more involved.

Example 18. *Find the volume of the solid formed by revolving the region bounded by $y^2 = x$ and $y = x^3$ about the (i) x-axis, (ii) y-axis, and (iii) the $x = -1$ axis.*

Notice beforehand that the intersection occurs at $(0, 0)$ and $(1, 1)$. So,

(i) *Revolving about the x-axis,*

$$V = \int_0^1 \pi(\sqrt{x}^2 - (x^3)^2)dx = \pi\left[\frac{x^2}{2} - \frac{x^7}{7}\right]_0^1 = \frac{5}{14}\pi$$

(ii) *About the y-axis,*

$$V = \int_0^1 \pi\left((\sqrt[3]{y})^2 - (y^2)^2\right)dy = \pi\left[\frac{3}{5}y^{5/3} - \frac{y^5}{5}\right]_0^1 = \frac{2}{5}\pi$$

(iii) *About $x=-1$ axis, we need to readjust the integrand.*

$$
\begin{aligned}
V &= \int_0^1 \pi\left((\sqrt[3]{y}+1)^2-(y^2+1)^2\right)dy \\
&= \pi\int_0^1 \left(y^{2/3}+2y^{1/3}+1-y^4-2y^2-1\right)dy \\
&= \pi\left[\frac{2}{5}y^{5/2}+\frac{6}{4}y^{4/3}-\frac{y^5}{5}-\frac{2}{3}y^3\right]_0^1 \\
&= \frac{37}{30}\pi
\end{aligned}
$$

13.2. **Cylindrical Shell Method.** The volume of a shell is equal to the outer volume minus the inner volume. Mathematically if $r<R$ represent the inner and outer radius of the cylinder respectively, then

$$
\begin{aligned}
V &= \text{outer} - \text{inner} \\
&= \pi R^2 h - \pi r^2 h \\
&= \pi h(R^2-r^2) \\
&= 2\pi h\frac{R+r}{2}(R-r) \\
&= 2\pi h R^* t
\end{aligned}
$$

where $R^*=\frac{R+r}{2}$ is the average radius and $t=R-r$ is the thickness of the shell.
Steps for this method:

(1) Suppose we have a region bounded by $y=f(x)$ an suppose it is revolved about the y-axis,
(2) Subdivide the interval $[a,b]$ into n equal subintervals $\Delta x=\frac{b-a}{n}$,
(3) Let $V(x_i)$ represent the volume of the cylindrical shell with outer radius x_i, inner radius x_{i-1} and height $f(x_i)$,
(4) then we have a sequence $V(x_1),V(x_2),\ldots$ of volumes of shells which fill our solid so the approximate volume is

$$\sum_{i=1}^{n}V(x_i)=\sum_{i=1}^{n}2\pi f(x_i)x_i^*\Delta x,\ \text{where } x_i^*=\frac{x_i+x_{i-1}}{2}$$

which is approximately

$$\sum_{i=1}^{n}2\pi x_i f(x_i)\Delta x$$

a Riemann sum! Then the volume is given by

$$V=\lim_{n\to\infty}\sum_{i=1}^{n}2\pi x_i f(x_i)\Delta x=2\pi\int_a^b xf(x)dx \tag{7}$$

Note: This is most often used when there is a hole punched in the solid.

Example 19. *Find the volume of the solid that remains after a hole of radius a has been bored through the centre of a sphere of radius $b(>a)$.*

Solution:

$$
\begin{aligned}
V &= 2\int_a^b 2\pi x\sqrt{b^2-a^2}dx, \qquad \textit{let } x = b\sin\theta, dx = b\cos\theta \\
&= 4\pi\int_{x=a}^{x=b} b^2\sin\theta\sqrt{b^2(1-\cos^2\theta)}b\cos\theta d\theta \\
&= 4\pi b^3\int_{x=a}^{x=b} \sin\theta\cos^2\theta d\theta, u=\cos\theta \\
&= 4\pi b^3\int_{x=a}^{x=b} u^2 du \\
&= 4\pi b^3\left[\frac{u^3}{3}\right]_{x=a}^{x=b}
\end{aligned}
$$

Now, with $u = \cos\theta = \frac{\sqrt{b^2-x^2}}{b}$ *we have*

$$4\pi b^3\left[\frac{u^3}{3}\right]_{x=a}^{x=b} = 4\pi b^3\left[\frac{(b^2-x^2)^{3/2}}{b^3}\right]_{x=a}^{x=b} = \frac{4}{3}\pi(b^2-a^2)^{3/2}$$

Example 20. *Using the method of cylindrical shells, find the volume obtained by revolving the region bounded by* $y = x^2$ *and* $y^2 = x$ *about the* y*-axis.*

Solution:

$$V = \int_0^1 2\pi x\left(\sqrt{x}-x^3\right)dx = 2\pi\int_0^1 (x^{3/2}-x^4)dx = 2\pi\left[\frac{2}{5}x^{5/2}-\frac{x^5}{5}\right]_0^1 = \frac{2}{5}\pi$$

In general we have obtained the following two formulas

$$\boxed{V = \int_a^b \pi f(x)dx \text{ and } V = \int_a^b 2\pi x f(x)dx}$$

14. Arc Length of a Parameterized Curve

Let $\mathcal{C}$ be the smooth curve obtained obtained by graphing $y = f(x)$ over the interval $[a,b)$. For the arc length of $\mathcal{C}$ we apply the following simple recipe.

(1) Subdivide the interval (a,b) into n equal subintervals of length $\Delta x = \frac{b-a}{x}$
(2) Let $a = x_0, \ldots, b = x_n$
(3) Let p_i be the point $(x_i, f(x_i))$ on $\mathcal{C}$ and $d(p_{i-1}, p_i)$ be the Euclidean distance between the points p_{i-1} and p_i. That is,

$$d(p_{i-1},p_i) = \sqrt{(x_i-x_{i-1})^2+(f(x_i)-f(x_{i-1}))^2} = (x_i-x_{i-1})\sqrt{1+\left(\frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}\right)^2} = \Delta x\sqrt{1+f'(x_i^*)^2}$$

where the last equality holds by the *Mean Value Theorem.*

Hence, now the approximate length of $\mathcal{C}$ is given by

$$\sum_{i=1}^n \sqrt{1+f'(x_i^*)^2}\Delta x$$

another Riemann sum! Finally, the *arc length* of $\mathcal{C}$ is given by

$$\boxed{\mathcal{L}ength(\mathcal{C}) = \lim_{n\to\infty}\sum_{i=1}^n \sqrt{1+f'(x_i^*)^2}\Delta x = \int_a^b \sqrt{1+f'(x)^2}dx} \tag{8}$$

Example 21. *Find the length of the curve* $\mathcal{C}$ *defined by* $y = x^{3/2}$ *on the interval* $[0,5]$.

Solution:

$$\begin{aligned}\mathcal{L}ength(\mathcal{C}) &= \int_0^5 \sqrt{1+y\prime(x)^2}dx \\ &= \int_0^5 \sqrt{1+\tfrac{9}{4}x}dx \\ &= \frac{1}{2}\int_0^5 \sqrt{4+9x}dx = \frac{335}{27}\end{aligned}$$

15. Numerical Approximation

This is used to approximate definite integrals when they cannot be integrated.
There are 3 common methods:

(1) *Midpoint Rule:* Given $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x$, recall that we had a choice of sample points x_i in the interval (x_{i-1}, x_i) and we have just been using the right hand side of the interval. Now, consider

$$R_n = \sum_{i=1}^n f(x_i)\Delta x \quad - \quad \text{The right-hand approximation}$$

$$L_n = \sum_{i=1}^n f(x_{i-1})\Delta x \quad - \quad \text{The left-hand approximation}$$

and finally

$$\boxed{M_n = \sum_{i=1}^n f(m_i)\Delta x \quad - \quad \text{The Midpoint approximation}}$$

where $m_i = \frac{x_{i-1}+x_i}{2}$

(2) *Trapezoidal Rule:* Here we use

$$\boxed{T_n = \frac{L_n+R_n}{2} = \frac{1}{2}\left[\sum_{i=1}^n f(x_{i-1})\Delta x + \sum_{i=1}^n f(x_i)\Delta x\right] = \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)\right]}$$

(3) *Simpson's Rule:* Here we use $2n$ sample points $x_1, x_2, \ldots, x_n, \ldots, x_{2n}$, but our intervals remain of size $\Delta x = \frac{b-a}{n}$

$$\boxed{S_{2n} := \frac{1}{3}\left[2M_n + T_n\right]}$$

Now,

$$M_n = 2\Delta x\left[\sum_{i=1}^n f(x_{2i-1})\right]$$

because midpoints here are the odd indexed sample points and

$$T_{2n} = 2\frac{\Delta x}{2}\left(f(x_0)+2f(x_2)+\ldots 2f(x_{2n-2})+f(x_{2n})\right)$$

So,

$$\begin{aligned}S_{2n} &= \frac{\Delta x}{3}\left[4\sum_{i=1}^n f(x_{2i-1}) + \left(f(x_0)+2f(x_2)+\ldots 2f(x_{2n-2})+f(x_{2n})\right)\right] \\ &= \frac{\Delta x}{3}\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+4f(x_{2n-1})+f(x_{2n})\right)\end{aligned}$$

Example 22. *Approximate the definite integral $\int_0^3 x^2 dx$ with $n = 6$ (i.e. so that $\Delta x = \frac{3-0}{6} = .5$ using; (i) midpoint rule, (ii) trapezoid rule, (iii) Simpson's rule.*
Solution:

(i) *For this, we need the following table of values:*

n	m_i	$f(m_i)$
1	.25	0.0625
2	.75	0.5625
3	1.25	1.5625
4	1.75	3.0625
5	2.25	5.0625
6	2.75	7.5625

So,

$$M_n = \sum_{i=1}^{6} f(m_i)\Delta x = \frac{1}{2}(19.875) = 8.9375$$

(ii)

$$\begin{aligned} T_n &= \frac{\Delta x}{2}(f(0) + 2f(0.5) + 2f(1) + + \cdots + 2f(5.5) + f(6)) \\ &= \frac{1}{4}(0^2 + 2(0.5)^2 + \cdots + 3^2) \\ &= 9.125 \end{aligned}$$

(iii)

$$S_n = \frac{\Delta x}{3}(0^2 + 4(0.5)^2 + 2(1)^2 + 4(1.5)^2 + 2(2)^2 + 4(2.5)^2 + 3^2) = \frac{1}{6}(54) = 9$$

15.1. **Error bounds on approximations.** Here we will just provide the error bounds without examining the required analysis.

For M_n; the error bound is

$$\boxed{|E(M_n)| \leq \frac{k(b-a)^3}{24n^2}}$$

where $k = \max_{x\in(a,b)}\{f''(x)\}$.

For T_n: Error is

$$\boxed{|E(T_n)| \leq \frac{k(b-a)^3}{12n^2}, \qquad k = \max_{x\in(a,b)}\{f''(x)\}}$$

For S_n; Error is

$$\boxed{|E(S_n)| \leq \frac{k(b-a)^5}{180n^4}, \qquad k = \max_{x\in(a,b)}\{f^{(4)}(x)\}.}$$

Example 23. *Estimate the error bound when using T_{10} or S_{10} approximation for $\int_1^2 \frac{1}{x}dx$.*
Solution: *Note that*

$$\frac{d^2}{dx^2}\left(\frac{1}{x}\right) = \frac{2}{x^3}$$

and

$$\frac{d^4}{dx^4}\left(\frac{1}{x}\right) = \frac{d^2}{dx^2}\left(\frac{2}{x^3}\right) = \frac{d}{dx}\left(\frac{-6}{x^4}\right) = \frac{24}{x^5}$$

so that the constant k for T_{10} is

$$k_T = \max_{[1,2]}\left(\frac{2}{x^3}\right) = 2$$

and for S_{10} is

$$k_S = \max_{[1,2]}\left(\frac{24}{x^5}\right) = 24.$$

Now,

$$|E(T_{10})| \leq \frac{k_T(2-1)^3}{12(10)^2} = \frac{1}{600}$$

and

$$|E(S_{10})| \leq \frac{k_S(2-1)^5}{180(10)^4} = \frac{24}{1800000}.$$

16. Improper Integrals

The following integrals

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx \text{ OR } \int_{-\infty}^a f(x)dx = \lim_{b\to -\infty} f(x)dx$$

are called *improper*. Similarly,

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

is improper (provided, both limits exist).
Warning: Do not do $\int_{-\infty}^\infty f(x)dx = \lim_{t\to\infty} \int_{-t}^t f(x)dx$, since one may not converge.

Example 24. *Evaluate the improper integral* $\int_{-\infty}^\infty \frac{1}{1+x^2}dx$.
Solution:

$$\begin{aligned} \int_{-\infty}^\infty \frac{1}{1+x^2}dx &= \lim_{b\to-\infty} \int_b^0 \frac{1}{1+x^2}dx + \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}dx \\ &= \lim_{b\to-\infty} [\arctan x]_b^0 + \lim_{t\to\infty} [\arctan x]_0^t \\ &= \left(0 - \frac{-\pi}{2}\right) + \left(\frac{\pi}{2} - 0\right) \\ &= \pi \end{aligned}$$

16.1. **Improper integrals of another type.** Suppose the function f is defined only on the interval $(a, b]$ with no specified value at a. Then we can still define the definite integral 'improperly' as

$$\int_a^b f(x)dx = \lim_{t\to a^-} \int_t^b f(x)dx.$$

The same can be done when $f(x)$ is defined on $[a, b)$, or even if, say $f(x)$ is only defined on $[a, c) \cup (c, b]$ (with a hole in the middle of the interval), then

$$\int_a^b f(x)dx = \lim_{t\to c^-} \int_a^t f(x)dx + \lim_{t\to c^+} \int_t^b f(x)dx$$

Note that the above integral will only converge if both limits exist.

Example 25.

Proposition 1. *[Comparison test for improper integrals] Suppose that* $f(x) \geq g(x)$ *on* $[a, \infty)$*. If* $\int_a^\infty f(x)dx$ *converges, then* $\int_a^\infty g(x)dx$ *also converges.*

Furthermore, if $\int_a^\infty g(x)dx$ *diverges, then so does* $\int_a^\infty f(x)dx$.

Example 26. *Does* $\int_0^\infty e^{-x^2}dx$ *converge?*

Soloution: *We have $x^2 \geq x$ on $[1,\infty]$ so $e^{x^2} \geq e^x$ on $[1,\infty)$ or $\frac{1}{e^{x^2}} \leq \frac{1}{e^x}$. So*

$$\begin{aligned}\int_0^\infty e^{-x^2}dx &= \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx \\ &\leq \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x}dx \\ &= \int_0^1 e^{-x^2}dx + \lim_{t\to\infty}\int_1^t e^{-x}dx \\ &= \int_0^1 e^{-x^2}dx + \lim_{t\to\infty}\left[-e^{-t} - (-e^{-1})\right] \\ &= \int_0^1 e^{-x^2}dx + e < \infty\end{aligned}$$

so our integral converges by comparison test.

17. Differential Equations

A *differential equation* (DE) is an equation which contains a derivative expression. The *order* of a DE is the highest derivative which appears in the equation.

Example 27. *$xy''(x) = \sin x e^{y(x)}$ is order 2.*

Another, simpler, DE is $y'(x) = \sin x + e^x$. This is easy to solve (integrate both sides), $y(x) = -\cos x + e^x + C$ is a solution.

Lets see a more exciting example.

Example 28. *Let $P(t)$ be a population function (with t for time). After studying the population over time, and discovering a proportionality amongst the $P(t)$ and its derivative. That is to say $P'(t) = kP(t)$, for some constant k. The goal now, is to obtain a formula for $P(t)$. Rearranging slightly is*

$$\begin{aligned}\frac{P'(t)}{P(t)} = k \Rightarrow \int \frac{P'(t)}{P(t)}dt &= \int k dt \\ \Leftrightarrow \ln|P(t)| &= kt + C \\ e^{\ln P(t)} &= e^{kt+C} \\ P(t) &= e^{kt}e^C \\ P(t) &= C'e^{kt}\end{aligned}$$

for some positive constant C that is determined by specified initial conditions (such as $P(0) = 100$).

Example 29. *Verify that $y = x^2 \ln x$ is a solution to $x^2y'' - 3xy' + 4y = 0$ in the region $x > 0$.*
Solution: *With $y = x^2 \ln x$, we compute $y' = 2x\ln x + x$ and $y'' = 2\ln x + 2 + 1$. Substituting into the DE, we find*

$$\begin{aligned}x^2y'' - 3xy' + 4y &= x^2(2\ln x + 3) - 3x(2x\ln x + x) + 4x^2\ln x \\ &= 2x^2\ln x + 3x^3 - 6x^2\ln x - 3x^2 + 4x^2\ln x \\ &= 0\end{aligned}$$

A *particular solution* to a DE is one without arbitrary constants. That is, any particular function satisfying the DE. The solution $y = x^2\ln x$ from the previous is example is a particular solution.

Example 30. *Suppose we want a solution to $y' + 2y = e^{-x}$ with $y(0) = 3$ [This is called an* initial value problem*].*

Check that $y = e^{-x} + Ce^{-2x}$ is a general solution. Indeed, $y' = -e^{-x} - 2Ce^{-2x}$ and

$$-e^{-x} - 2Ce^{-2x} + 2(e^{-x} + Ce^{-2x} = 0.$$

Now since $y(0) = 3$, we can solve for the constant C as

$$3 = e^0 + Ce^0 = 1 + C \Rightarrow C = 2$$

so that $y = e^{-x} + 2e^{-2x}$ is a particular solution to the initial value problem (IVP).

17.1. **Separable Differential Equations.** If a DE can be expressed in the form $y' = g(x)f(y)$, we say it is *separable.* Equivalently

$$\frac{1}{f(y)}\frac{dy}{dx} = g(x) \Leftrightarrow \frac{1}{f(y)}dy = g(x)dx.$$

A separable DE is solved by integrating both sides of this rearrangement.

$$\int \frac{1}{f(y)}dy = \int g(x)dx,$$

but this requires some justification: If we are given a DE of the form $N(y)dy = M(x)dx$ or equivalently $N(y)\frac{dy}{dx} - M(x) = 0$. Now let $H_1(y)$ be an anti-derivative of $N(y)$ and $H_2(x)$ an anti-derivative of $M(x)$. This equation is then equivalent to the following:

$$\begin{aligned} &H_1'(y)\frac{dy}{dx} - H_2'(x) = 0 \\ &\frac{d}{dx}(H_1(y)) - H_2'(x) = 0 \\ &= \frac{d}{dx}\left(H_1(y) - H_2(x)\right)\right) = 0 \end{aligned}$$

Now, this implies that $H_1(y) - H_2(x) = K$ (is constant) and we can find $y(x)$ from $H_2(y)$.

Example 31. *Is $\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$ separable? If so, solve it.*
Solution: *Yes, it is solvable and it can be re-expressed as*

$$2(y-1)dy = (3x^2 - 4x + 2)dx \Rightarrow 2\int (y-1)dy = \int (3x^2 - 4x + 2)dx.$$

This integrates to

$$y^2 - 2y = x^3 + 2x^2 + 2x + C.$$

Now, although this is separable an we managed to integrate it, we find that the solution cannot be explicitly expressed as $y = \cdots$. In attempt to extract an explicit solution, we treat this as a quadratic equation in y.

$$y^2 - 2y - (x^3 + 2x^2 + 2x + C) = 0$$

which has solutions,

$$y = \frac{2 \pm \sqrt{4 - 4(-1)(x^3 + 2x^2 + 2x + C)}}{2} = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 1 + C}.$$

If we had been given the IVP of $y(0) = -1$, then this would imply that the solution be $y = 1 - \sqrt{x^3 + 2x^2 + 2x + 1 + C}$.

17.2. **Linear Differential Equations.** A *linear differential equation* is most generally expressed as

(9) $$y' + p(x)y = q(x)$$

Theorem 6. *The solution to the linear differential equation* (9) *is*

$$y = \frac{1}{I(x)}\int I(x)q(x)dx$$

where $I(x) = e^{\int p(x)dx}$.

This function $I(x)$ is called the integrating factor *and is a function satisfying that*

$$I(x)\left(y' + p(x)y\right) = \left(I(x)y\right)'.$$

Proof. First, verifying that $I(x) = e^{\int p(x)dx}$ satisfies the above claimed equation. Indeed, expanding the right by the product rule, we have

$$I(x)\left(y' + p(x)y\right) = (I(x)y)' = I'(x)y + I(x)y' \Leftrightarrow I(x)p(x)y = I'(x)y$$

so that, provided $I(x) \neq 0$, $p(x) = \frac{I'(x)}{I(x)}$. Integrating this says that

$$\int p(x)dx = \int \frac{I'(x)}{I(x)}dx \Leftrightarrow \ln|I(x)| = \int p(x)dx \Leftrightarrow I(x) = e^{\int p(x)dx}.$$

Now, that this holds, the left hand side of our DE is expressed as $(I(x)y)'$ and now reads, $(I(x)y) = q(x)$. Hence, integrating this, $I(x)y = \int q(x)dx$ so $y = \frac{1}{I(x)}\int I(x)q(x)dx$ as claimed. □

Note that this method of solving a DE requires it be of first order (i.e. linear).

Example 32. (1) *Solve the IVP $y' - 2xy = 1$ with $y(0) = 1$.*
Solution: *The integrating factor here is $I(x) = e^{-\int 2xdx} = e^{-x^2}$, so*

$$y = \frac{1}{e^{-x^2}}\int e^{-x^2}dx = e^{x^2}\left(\int_0^x e^{-t^2}dt + C\right) = e^{x^2}\int_a^x e^{-t^2}dt + Ce^{x^2}$$

and with $y(0) = 1$ we find C as

$$1 = e^0\int_0^0 e^{-t^2}dt + e^0 C \Rightarrow C = 1$$

(2) *Solve the IVP $(y - x\sin x)dx + xdy = 0$ with $y(\pi) = 3$.*
Solution: *This is equivalent to $y = x\sin x + xy' = 0$ or, better, $y' + \frac{1}{x}y = \sin x$. For the integrating factor $I(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x|$. Now,*

$$\begin{aligned} y &= \frac{1}{|x|}\int |x|\sin x dx, \\ &= \frac{1}{x}\left[-x\cos x + \int \cos x dx\right], \qquad \textit{IBP with } u = x, dv = \sin x dx \\ &= -\cos x + \frac{\sin x}{x} + \frac{C}{x} \end{aligned}$$

18. Sequences and Series

Recall, the limit of a sequence: A sequence $\{x_n\}$ converges to L if for all $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $|x_n - L| < \epsilon$ for all $n > k$.

Example 33. $a_n = (-1)^n \frac{\cos n}{n^2}$ *and* $\left|(-1)^n\frac{\cos n}{n^2}\right| \leq \frac{1}{n^2} \to 0$ *and by* squeeze theorem, $a_n \to 0$.

An *increasing (decreasing) sequence* is one such that $a_{n+1} \geq (\leq) a_n$ for all $n \geq 1$. A *monotonic sequence* is one which is either strictly increasing or decreasing.

Theorem 7 (Monotone sequence theorem)**.** *A bounded monotonic sequence always has a limit (i.e. will always converge).*

18.1. **Series.** Let a_n be a sequence, then $\sum_{n=1}^{\infty} = a_1 + a_2 + a_3 + \ldots$ is an *infinite series*. The n^{th} *partial sum* of a series is $S_n = \sum_{k=1}^{n} a_k$. To evaluate an infinite series we use the limit of the partial sums:

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\sum_{k=1}^{n} a_k.$$

If the limit exists, series converges. If limit does not exist, series *diverges*.
Warning: When discussing series, say $\sum_{n=1}^{\infty} c_n$, there are two sequences in play. We have the sequence itself c_n and the partial sums S_n, so remember

$$\sum_{n=1}^{\infty} = \lim_{n\to\infty} S_n.$$

Example 34.

$$c_i = i \Rightarrow \sum n = 1^\infty n = \lim_{n\to\infty} \sum_{k=1}^{n} k = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty, \text{ (diverges)}$$

Consider the *geometric series*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^n + \ldots .$$

The partial sums are given by

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and multiplying by r says,

$$rS_n = ar + ar^2 + \cdots + ar^n.$$

Subtracting these, we have

$$S_n - rS_n = a - ar^n \Leftrightarrow S_n(1-r) = a(1-r^n)$$

so that

$$S_n = \frac{1(1-r^n)}{1-r}.$$

Now, when does a geometric series converge? We must have that $\lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$ exists, which means, in order for a geometric series to converge we must have $|r| < 1$.

Furthermore, if $|r| < 1$, then the limit can be computed exactly as $\lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$. That is, the sum a geometric series is

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \frac{a}{1-r}, |r| < 1 \\ \infty, |r| \geq 1 \end{cases}$$

Example 35. *Express the repeating decimal* $1.2\overline{31}$ *as an infinite series and then as a fraction.*
Solution: *We have*

$$1.2\overline{31} = 1.2 + \frac{31}{10^3} + \frac{31}{10^3} \cdot \frac{1}{10^2} + \frac{31}{10^3} \cdot \frac{1}{(10^2)^2} + \ldots$$

Letting $a = \frac{31}{10^3}$ *and* $r = \frac{1}{10^2} < 1$, *so that*

$$1.2\overline{31} = 1.2 + \sum_{n=1}^{\infty} ar^{n-1} = \frac{12}{10} + \frac{31}{10^3} \cdot \frac{1}{1 - \frac{1}{10^2}} = \frac{1219}{990}.$$

Theorem 8. *Suppose* $\sum_{n=1}^{\infty} c_n = L$ *and* $\sum_{n=1}^{\infty} b_n = M$ *and let* $k \in \mathbb{R}$. *Then,*

$$\sum_{n=1}^{\infty} kc_n = kL$$

and

$$\sum_{n=1}^{\infty} (c_n + b_n) = L + M.$$

Note that this theorem requires that both converge.

Theorem 9. *If the series* $\sum_{n=1}^{\infty} c_n$ *converges, then the sequence* c_n *must converge to 0.*

Proof. Given that $\sum_{n=1}^{\infty} c_n = L$, consider the sequence S_n of partial sums. Observe, that for each n, $c_n = s_n - S_{n-1}$ and now,

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} S_n - S_{n-1} = L - L = 0.$$

□

18.2. **Divergence test.** Suppose that $\sum_{n=1}^{\infty} c_n$ is a series. If $\lim_{n\to\infty} c_n \neq 0$ then the series diverges. This is just the converse of the previous result. Otherwise, divergence test provides no information.

Example 36. *Discuss the convergence/divergence of the series*

(1) $\sum_{n=1}^{\infty} \left(\frac{3}{n} + \frac{5}{4^n}\right)$.
Solution:

$$\sum_{n=1}^{\infty} \left(\frac{3}{n} + \frac{5}{4^n}\right) = \sum_{n=1}^{\infty} \frac{3}{n} + \sum_{n=1}^{\infty} \frac{5}{4^n}$$
$$= 3 \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{diverges} + \underbrace{\sum_{n=1}^{\infty} \frac{5}{4}\left(\frac{1}{4}\right)^{n-1}}_{converges (geometric\, r=1/4)}$$

Therefore, the series diverges.

(2) $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)}$
Solution: *By partial fractions, we find*

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)} = \sum_{n=1}^{\infty} \left(\frac{-1}{2(2n+1)} + \frac{1}{2(2n-1)}\right)$$
$$= \lim_{n\to\infty} -\frac{1}{6} + \frac{1}{2} - \frac{1}{10} + \frac{1}{6} - \frac{1}{14} + \frac{1}{10} + \cdots + \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$
$$= \lim_{n\to\infty} \frac{1}{2} - \frac{1}{2(2n-1)} = \frac{1}{2}$$

This is a telescoping series*.*

Theorem 10 (Integral test for convergence of a series)**.** *Let $\sum_{n=1}^{\infty} c_n$ be a series and suppose $f(x)$ is a function on $[1,\infty)$ such that*

(1) *$f(x) > 0$ on $[1,\infty)$.*
(2) *$f(x)$ is continuous on $[1,\infty)$*
(3) *$f(x)$ is non-increasing on $[1,\infty)$ (i.e. $f'(x) \leq 0$)*
(4) *$f(n) = c_n$ for all n.*

Then,

$$\sum_{n=1}^{\infty} c_n \text{ converges } \Leftrightarrow \int_1^{\infty} f(x)dx \text{ converges.}$$

Proof. Suppose that $\sum_{n=1}^{\infty} c_n$ converges and show that $\int_1^{\infty} f(x)dx$ converges. Certainly

$$S_n = c_1 + c_2 + \cdots + c_n \geq \int_1^n f(x)dx$$

so

$$S = \lim_{n\to\infty} S_n \geq \int_1^{\infty} f(x)dx$$

which implies convergence.

Suppose now that $\int_1^{\infty} f(x)dx$ converges and show the converse. Observe that

$$c_2 + c_3 + \cdots + c_n \leq \int_1^{\infty} f(x)dx$$

so

$$S_n = c_1 + c_2 + \ldots c_n \leq c_1 + \int_1^{\infty} f(x)dx$$

and hence $\lim_{n\to\infty} S_n \leq c_1 + \int_1^{\infty} f(x)dx < \infty$ so converges. ☐

18.3. **Estimating the value of a series.** Suppose that $\sum_{n=1}^{\infty} c_n$ is a series proved to be convergent by the integral test (say $\sum_{n=1}^{\infty} c_n = S$).

We would like to approximate S by the partial sums S_n and determine what the error will be. Let $R_n = S - S_n$ be the *remainder* (Note that, we are assuming that our c_n's are positive so that the partial sums are an increasing sequence). So, also

$$R_n = S - S_n = (c_1 + c_2 + \cdots + c_n + \dots) - (c_1 + c_2 + \cdots + c_n) = c_{n+1} + c_{n+2} + \dots$$

and we claim that

$$\int_{n+1}^{\infty} f(x)dx \le R_n \le \int_{n}^{\infty} f(x)dx$$

for $f(n) = c_n$.

First, it is easy to see that $R_n = c_{n+1} + c_{n+2} + \cdots \le \int_n^{\infty} f(x)dx$. Now, also $R_n \ge \int_{n+1}^{\infty} f(x)dx$ since these rectangles are an over approximation of the integral.

Now,

$$\begin{aligned}\int_{n+1}^{\infty} f(x)dx \le& R_n \le \int_{n}^{\infty} f(x)dx\\ S_n + \int_{n+1}^{\infty} f(x)dx \le& S_n + R_n \le S_n + \int_{n}^{\infty} f(x)dx\\ S_n + \int_{n+1}^{\infty} f(x)dx \le& S \le S_n + \int_{n}^{\infty} f(x)dx\end{aligned}$$

So we know that S lies in the interval $\left[S_n + \int_{n+1}^{\infty} f(x)dx, S_n + \int_n^{\infty} f(x)dx\right]$ and the error cannot be more than the length of the interval. That is,

$$\boxed{|\text{Error}| \le (S_n + \int_{n}^{\infty} f(x)dx) - (S_n + \int_{n+1}^{\infty} f(x)dx) = \int_{n}^{\infty} f(x)dx - \int_{n+1}^{\infty} f(x)dx = \int_{n}^{n+1} f(x)dx}$$

Example 37. *Use S_{10} to help calculate $\sum_{n=1}^{\infty} \frac{1}{n^4}$.*

Solution: *With a calculator we find $S_{10} = 1.082036583$. Notice that $\frac{1}{x^4}$ is continuous, decreasing on $[1, \infty)$ and matches the values of the sequence. So, a good approximation is the midpoint of the interval $\left[S_{10} + \int_{11}^{\infty} \frac{1}{x^4} dx, S_{10} + \int_{10}^{\infty} \frac{1}{x^4} dx\right]$. That is,*

$$S \approx \frac{2S_{10} + \int_{11}^{\infty} \frac{1}{x^4} dx + \int_{10}^{\infty} \frac{1}{x^4} dx}{2} \approx 1.082328469$$

18.4. **p-series.** Any series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where p is a fixed positive number.

Theorem 11. *If $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a p-series, then it converges if and only if $p > 1$.*

Proof. We first consider the special case that $p = 1$. This series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the *Harmonic series.* By the integral test $\int_1^{\infty} \frac{1}{x} dx \lim_{t\to\infty} \int_1^t \frac{1}{x} dx = \lim_{t\to\infty} \ln x|_1^t = \infty$. This diverges and hence, the harmonic series diverges.

Now, consider the case $p \neq 1$:

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim t \to \infty \left[\frac{x^{1-p}}{1-p}\right]_1^t = \frac{1}{1-p} \lim_{t\to\infty} \frac{1}{t^{p-1}} - 1$$

which diverges if $p - 1 < 0\,(p < 1)$ and converges if $p - 1 > 0\,(p > 1)$. □

18.5. **Comparison test.**

Theorem 12 (Comparison test)**.** *Let $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ be two series with $0 \le a_n \le b_n$. Then*

(1) $\sum_{n=1}^{\infty} a_n$ *converges if* $\sum_{n=1}^{\infty} b_n$ *converges*
(2) $\sum_{n=1}^{\infty} b_n$ *diverges if* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof. Let A_n and B_n represent the n^{th} partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ respectively. Suppose that $B_n \to L$. Then since $0 \leq a_n \leq b_n$ we have $0 \leq A_n \leq B_n$ for all $n > 0$. Now, A_n and B_n are both monotone increasing sequences and with $B_n \to L$ we have A_n is bounded above by L. By monotone sequence theorem, A_n converges.

For the second part, A_n is monotone increasing to infinity. Thus, also $B_n \geq A_n$ is also increasing to infinity. □

18.6. **Limit comparison test.**

Theorem 13 (Limit comparison test)**.** *Suppose there are two series $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ with both $a_n, b_n > 0$ for all n. Then*

(1) *If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0, \text{ or } \infty$$

then either both converge or both diverge.

(2) *If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

and $\sum b_n$ converges, then so does $\sum a_n$.

(3) *If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$$

and $\sum b_n$ diverges, then so does $\sum a_n$.

Corollary 1. *If $\sum_{n=1}^{\infty} |a_n|$ converges then so does $\sum_{n=1}^{\infty} a_n$.*

Proof. Suppose that $\sum_{n=1}^{\infty} |a_n|$ converges to M. We have the inequality

$$0 \leq a_n + |a_n| \leq 2|a_n|.$$

By the comparison test, then

$$\sum_{n=1}^{\infty}(a_n + |a_n|)$$

converges (since $\sum 2|a_n| = 2\sum |a_n| = 2M$). Assuming that $\sum(a_n + |a_n|) = L$ then $\sum a_n = L - M$ as required. □

Definition 4. *If $\sum a_n$ is a series such that $\sum |a_n|$ converges, we say that $\sum a_n$ is* absolutely convergent*.*

If $\sum a_n$ converges, but $\sum |a_n|$ does not, then $\sum a_n$ is said to be conditionally convergent*.*

Any series of the form $\sum_n (-1)^n c_n$ where c_n's are positive, then the series is called an alternating series*.*

Theorem 14 (Alternating series test)**.** *Let $\sum_1^{\infty} (-1)^{n-1} c_n$ be an alternating series. If $c_{n+1} \leq c_n$ for all $n > 0$ and $\lim_{n\to\infty} c_n = 0$ then the series converges.*

Proof. Supposing that $\lim_{n\to\infty} c_n = 0$ and $0 \leq c_{n+1} \leq c_n$, let S_n denote the partial sums of the series $\sum_n (-1)^n c_n$. Note that S_{2n} is monotone increasing and bounded above by c_1. So it remains to show that the limit of S_{2n} is the same as the sum of the series. Note that $S_{2n+1} = S_{2n} + c_{2n+1} \to S + 0 = S$ so thus, the series converges. □

18.7. **Alternating series estimation.**

Theorem 15 (Alternating series estimation theorem)**.** *If $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ is the sum of an alternating series which satisfies*

$$0 \leq b_{n+1} \leq b_n \text{ and } \lim_{n\to\infty} b_n = 0$$

then,

$$R_n \leq |S - S - n| \leq b_{n+1}.$$

Proof. We know from above that S lies between two consecutive partial sums S_n and S_{n+1}. It follows that $|S - S_n| \leq |S_{n+1} - S_n| = b_{n+1}$. □

Note; the rule that the error is smaller than the first neglected term is valid only for alternating series.

18.8. **Ratio test.** Let $\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ for a series $\sum_n a_n$. Then
if $\rho < 1$, the series is absolutely convergent and hence convergent
if $\rho > 1$ or $= \infty$ then the series is divergent
if $\rho = 1$ or does not exist then the test is inconclusive.

18.9. **Root test.** Let $\sum_n a_n$ be a series. Then
if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$ then $\sum a_n$ is absolutely convergent,
if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $= \infty$ the series is divergent
and otherwise, if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$ the test is inconcolsive

19. Series testing strategies

The following is a recipe for testing convergence of series.

(1) Check if $\sum a_n$ is geometric, harmonic, p-series or telescoping
(2) Is the $\lim_{n\to\infty} a_n = 0$? If not, then the series diverges (by divergence test). Otherwise, convergence is still unknown
(3) Does $|a_k|$ resemble r^k or $\frac{1}{k^p}$ for large k? If so, try the comparison test, or limit comparison test with geometric series or a p-series
(4) If $a_k = f(k)$ for some positive, decreasing continuous function f, try the integral test
(5) If there are powers of n or $n!$ the ratio test is probably a good idea.
  - This test may result in the series diverges with $\rho > 1$ or $= \infty$ so there would be no need to proceed with alternating series.
  - This may show that $\sum_n a_n$ converges absolutely (when $\rho < 1$) hence converges
  - if $\rho = 1$ or doesn't exist, this test has failed and says nothing
(6) Try the Root test, especially if you see powers of n!
  - If $\rho > 1$ or $= \infty$ series diverges
  - if $\rho < 1$ it converges absolutely and hence converges
  - if $\rho = 1$, or does not exist the try something else because we still don't know
(7) if both ratio and root test have already failed, or it is clear that $\sum |a_n|$ is divergent, try (if series is alternating) the alternating series test incase it is conditionally convergent.
(8) Error bounds:
  - if series converges by the integral test, use the integral formula for approximate value of series.
  - if the series converges by the alternating series test, use the alternating series estimation theorem to approximate the value.

20. Power series

A *power series* centered at a is a series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$ with x a variable. This series differs in values because of the variable x. This means that certain values cause series to converge and some won't. This should give the impression that a power series is a function whose domain is a collection of values x for which the series converges. When studying these, we will be interested in finding an elementary function f such that $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ as well as determining its domain.
Observations: We already know that $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \dots$ converges to $\frac{1}{1-r}$ for $|r| < 1$ while it diverges everywhere else. So, we could say that $\sum_{n\geq 0} x^n$, centered at zero converges to the function $\frac{1}{1-x}$ on the interval $(-1, 1)$ and diverges elsewhere.

The function $T_n(x) := \sum_{k=0}^{n} c_n(x-a)^k$ is a polynomial. Recall, from Taylor polynomials that if $T_1(x) = 1, T_2(x) = 1 + x, \dots, T_k(x) = 1 + x + x^2 + \dots + x^k$ that $\lim_{n\to\infty} T_n(x) = \frac{1}{1-x}$. Thus, a power series can be viewed as a function whose curve is a limit of a sequence of polynomials.

Also notice that a power series $\sum_n c_n(x-a)^n$ always converges for $x = a$.

Here, we will mainly discuss two things;

(1) For what values of x does a power series converge? (this is called the *radius and interval of convergence*
(2) To what function, if any, does it converge to?

Theorem 16. *The power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ converges either for $x = a$, for all x or there is a positive number R such that the series converges for all $|x-a| < R$ and diverges for all $|x-a| > R$.*

Note: *When a power series is known to converge for all $|x-a| < R$, one must manually determine convergence at the endpoints of this open interval.*

Proof. Given the power series $\sum_{k=0}^{\infty} c_k(x-a)^k$, suppose we define $\rho = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$. Applying the ratio test to $\lim_{n\to\infty} \left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = \rho|x-a|$. Then this converges for $\rho|x-a| < 1$ or, equivalently, $|x-a| < 1/\rho$. So to say that $R = 1/\rho$. □

Remark 2. *The above theorem states convergence on intervals centered at a. This is called the* interval of convergence. *Series may or may not converge at the endpoints of this interval. The $R = 1/\rho = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$. If the power series converges to a function, we say $f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$ for all $|x-a| < R$*

Example 38. *Discuss the convergence of the following power series*

(1)
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n4^n}(x-2)^n$$

Solution: *Here*
$$R = 1/\rho = \lim_{n\to\infty} \left|\frac{(-1)^n}{n4^n} \frac{(n+1)4^{n+1}}{(-1)^{n+1}}\right| = \lim_{n\to\infty} |4 + 4/n| = 4$$
so $|x-2| < 4$ or $-2 < x < 6$. At the endpoints; when $x = -2$ this is
$$\sum_{n=1}^{\infty} \frac{(-1)^n(-4)^n}{n4^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$
when $x = 6$ this becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is known to converge! Thus, the interval of convergence is $(-2, 6]$.

(2)
$$\sum_{n=1}^{\infty} \frac{2^n}{n!} x^n$$

Solution: *Here we find $\rho = \lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n}\right| = \lim_{n\to\infty} \frac{2}{n+1} = 0$ hence $1/\rho = \infty$ which implies radius of convergence is infinite and series converges for all values of x.*

20.1. **Functions as power series.** Using that $\frac{1}{1-x} = \sum_{k\geq 0} x^k$ as a basis, we can find other functions expressed as power series!

For example,

(1)
$$\frac{1}{3+x} = \frac{1}{3}\frac{1}{1-(-\frac{x}{3})} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+1}}$$
and the interval of convergence is $\left|\frac{x}{3}\right| < 1$ or, $|x| < 3$.

(2)
$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1)^n x^2 n$$
with interval of convergence $x^2 < 1$ which is really just $|x| < 1$.

(3)
$$\frac{x^5}{3+x^2} = \frac{x^5}{3}\frac{1}{1-(-\frac{x^2}{3})} = \frac{x^5}{3}\sum_{n=0}^{\infty} \frac{(-1)^n x^2 n}{3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+5}}{3^{n+1}}$$
with radius of convergence $\left|\frac{x^2}{3}\right| < 1$ or $|x| < \sqrt{3}$

Now that we can deal with everything of this form, we will need a new style.

Theorem 17. *Suppose* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has radius of convergence* $R > 0$ *and suppose that* $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$. *Then* $f(x)$ *is differentiable of* $|x-a| < R$ *and*

$$f\prime(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$\int_a^x f(x)dx = \int_a^x \sum_{n=0}^{\infty} c_n(x-a)^n dx = \sum_{n=0}^{\infty} \int_a^x c_n(x-a)^n dx = \sum_{n=0}^{\infty} \frac{c_n x^{n+1}}{n+1}$$

and both of these resulting power series have R *for their radius of convergence.*

Example 39. *Show that* $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$ *with* $R = 1$.
Solution:

$$\begin{aligned}\frac{d}{dx}(1-x)^{-1} &= \frac{1}{(1-x)^2} \\ &= \frac{d}{dx}\sum_{n=0}^{\infty} x^n \\ &= \sum_{n=1}^{\infty} nx^{n-1} \\ &= \sum_{n=0}^{\infty}(n+1)x^n\end{aligned}$$

$$\begin{aligned}\ln(1+x) &= \int_a^x \frac{1}{1+t}dt \\ &= \int_a^x \frac{1}{1-(-t)}dt \\ &= \int_0^x \sum_{n=0}^{\infty}(-1)^n t^n dt \\ &= \sum_{n=0}^{\infty}(-1)^n \int_0^x t^n dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}\end{aligned}$$

and radius of convergence $R = 1$.

$$\begin{aligned}\arctan x &= \int_0^x \frac{1}{1+x^2}dx \\ &= \int_0^x \frac{1}{1-(-x^2)}dx \\ &= \int_0^x \sum_{n=0}^{\infty}(-1)^n x^{2n} dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}\end{aligned}$$

for $|x| < 1$.

20.2. **Taylor and Mclaurin series.** Suppose $f(x)$ has power series representation $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$. What do the c_n's look like?

Theorem 18. *Let* $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ *on* $|x-a| < R$, *then*

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

That is to say

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Example 40. *We know that* $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ *which implies that*

$$\frac{(-1)^n}{2n+1} = \frac{\arctan^{(n)}(0)}{n!}$$

so the n^{th} *derivative of* arctan *evaluated at zero is* $\frac{(-1)^n n!}{2n+1}$.

Definition 5 (Taylor series)**.** *Let* $f(x)$ *be a function which is differentiable infinitely many times, then the series*

$$T_{f,a}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the Taylor series *generated by* f *centred at* a.

Any Taylor series centred at 0, is called a McLaurin series.

Example 41. (1) $\sum_{n=0}^{\infty} x^n$ *is the Mclaurin series for* $\frac{1}{1-x}$.

(2) *Find the McLaurin series generated by* $f(x) = e^x$ *and determine the interval of convergence.*

Solution: *We need only determine* $c_n = \frac{f^{(n)}(0)}{n!} = \frac{e^x}{n!}|_{x=0} = \frac{1}{n!}$. *So*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Notice that there exists functions which generate taylor and McLaurin series that do not converge to the original function. For example, the piecewise defined function

$$f(x) = \begin{cases} e^{-1/x^2}, x \neq 0 \\ 0, x = 0 \end{cases}$$

has a McLaurin series of 0, but this function is not the zero function.

The n^{th} *degree taylor polynomial* for f, centred at a is defined as the truncated version of its Taylor series $T_n(x) = \sum_{k=0}^{n} \frac{f^{(n)}(a)}{n!}(x-a)^n$. These are the partial sums of the Taylor series and again we can define the n^{th} *remainder* $R_n(x) = T_{f,a}(x) - T_n(x)$.

We have that

$$R_n(x) \to 0 \Leftrightarrow T_n(x) \to f(x)$$

To show that $R_n(x) \to 0$, we will use a tool called *Taylor's inequality.*

Theorem 19. *Let* $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ *be a Taylor series with radius of convergence* R. *Then* $|R_n(x)|$ *can be bounded above in the following way:*

If $|f^{(n+1)}(x)| \leq M_n$ *on* $|x-a| < R$, *then*

$$|R_n(x)| \leq \frac{M_n|x-a|^{n+1}}{(n+1)!}, \text{ on } |x-a| \leq d < R.$$

If we can show that $\lim_{n\to\infty} \frac{M_n|x-a|^{n+1}}{(n+1)!} = 0$ *for all* $|x-a| \leq d$, *then*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Example 42. *Recall, Mclaurin series of* $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ *has infinite radius of convergence (since* $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ *since* $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ *converges by divergence test).*

Show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ *converges to* e^x*.*

Solution: *Consider any interval* $[-k,k]$*. Since* e^x *is increasing let* $M_n = e^k$*. Then Taylors inequality guarantees that* $|R_n(x)| \leq \frac{e^k|x|^{n+1}}{(n+1)!}$*. Now,*

$$\lim_{n\to\infty} |R_n(x)| \leq e^k \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

as required.

Example 43. *Find the McLaurin series generated by* $\sin x$*.*

Solution: *since the derivatives of sine are* $\cos, -\sin, -\cos, \sin, \ldots$ *and repeat every four, we find the values of these derivates to be* $0, 1, 0, -1, \ldots$ *and repeat every four. Hence, then the series is given by*

$$0 + 1\cdot x + 0\frac{x^2}{2!} - 1\frac{x^3}{3!} + 0\frac{x^4}{4!} + \cdots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

and

$$R = \lim_{n\to\infty} \left|\frac{(2n+3)!}{(2n+1)!}\right| = \lim_{n\to\infty} (2n+3)(2n+2) = \infty$$

Does the series generated by $\sin x$ converge to $\sin x$?

We have

$$\sum_{n=0}^{\infty} \frac{\sin^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

and $|\sin^{(n)}(x) \leq 1 = M_n$ for all $x \in [-k,k]$. Thus,

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \to 0$$

So the Taylor series generated by $\sin x$ converges to $\sin x$ for all x.

Note that if you wanted to approximate $\sin x$ near $x = 50$, choose $a = 50$.

Example 44. *Estimate* $\sin(0.1)$ *with error* $< 10^{-8}$*.*

Solution: *Recall* $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ *and we know that* $|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$*. That is,*

$$\begin{aligned} |\sin x - T_n(x)| \leq & \frac{|x|^{n+1}}{(n+1)!} < 10^{-8} \\ & \frac{|0.1|^{n+1}}{(n+1)!} < 10^{-8} \end{aligned}$$

Try $n = 6$*, and find that* $R_6(0.1) \leq \frac{(0.1)^7}{(6+1)!} = 1.9 \times 10^{-11} < 10^{-8}$*. Thus,* $T_6(0.1) = T_5(0.1) = (0.1) - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!} = 0.099833$*.*

Example 45. *Find the McLaurin series for* $\cos x$*.*

Solution:

$$\cos x = \frac{d}{dx}\sin x = \frac{d}{dx}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Theorem 20. *Let* $\sum_n a_n x^n, \sum_n b_n x^n$ *be two McLaurin series with radii of convergence* R_a *and* R_b *respectively. Then*

$$\sum_n c_n x^n = \left(\sum_n a_n x^n\right) \cdot \left(\sum_n b_n x^n\right)$$

where

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{i=0}^{n} a_i b_{n-i}$$

has interval of convergence $R = \min\{R_a, R_b\}$.

Example 46. *Find the Mclaurin series for* $\frac{1}{x-1}\ln(1-x)$.
Solution:

$$\frac{1}{x-1} = \frac{-1}{1-x} = (-1)(1 + x + x^2 + \dots)$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

which implies

$$\begin{aligned}
\frac{1}{x-1}\ln(1-x) &= (1 + x + x^2 + \dots)(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots) \\
&= 1(0) + (1(0) + (1)(1))\,x + \left(1(0) + 1(\frac{1}{2}) + 1(1)\right) x^2 + (1 + 1/2 + 1/3)\,x^3 + \dots \\
&= \sum_{n=0}^{\infty} (1 + 1/2 + 1/3 + \cdots + 1/n)\,x^n
\end{aligned}$$

and $R = 1$.

Everyone should know the following list of Taylor expansions

$$\begin{aligned}
e^x &= \sum_{n\geq 0} \frac{x^n}{n!} \\
\cos x &= \sum_{n\geq 0} \frac{(-1)^n x^{2n}}{(2n)!} \\
\sin x &= \sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\
\arctan x &= \sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{2n+1} \\
\frac{1}{1-x} &= \sum_{n\geq 0} x^n \\
\ln(1-x) &= -\sum_{n\geq 0} \frac{x^n}{n}
\end{aligned}$$

21. Applications of Taylor and McLaurin series

(1) Find the sum of the series $1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \cdots = \sum_{k\geq 0} \frac{(-1)^k \pi^{2k}}{(2k)!} = \cos(\pi) = -1$.
(2) Approximate the integral $\int_0^1 \frac{1-\cos x}{x^2}dx$ with error less than 0.00001.

Solution:

$$\begin{aligned}
\int_0^1 \frac{1-\cos x}{x^2}dx &= \int_0^1 \frac{1}{x^2}\left(1-(1-x^2/2!+x^4/4!-x^6/6!+\dots\right)dx \\
&= \int_0^1 \frac{1}{x^2}\left(x^2/2!-x^4/4!+x^6/6!+\dots\right)dx \\
&= \int_0^1 \left(1/2!-x^2/4!+x^4/6!+\dots\right)dx \\
&= \left[\frac{x}{2}-\frac{x^3}{3\cdot 4!}+\frac{x^5}{5\cdot 6!}+\dots\right]_0^1 \\
&= \frac{1}{2!}-\frac{1}{3\cdot 4!}+\frac{1}{5\cdot 6!}-\frac{1}{7\cdot 8!}+\dots
\end{aligned}$$

This is an alternating series, so the error bound for S_n is c_{n+1}. Letting $n=4$, then $c_4=\frac{1}{7\cdot 8!}=0.0000036<0.00001$. Therefor S_3 is appropriate and

$$S_3=\frac{1}{2!}-\frac{1}{3\cdot 4!}+\frac{1}{5\cdot 6!}\approx 0.486388$$

(3) Approximate the integral $\int_0^1 \frac{1-\cos x}{x^2}dx$ with error less than 10^{-7}.
Solution:

$$\left|\sin x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\right)\right|\le |R_n(x)|$$

implies

$$\left|\frac{\sin x}{x}-\left(\frac{1}{x}\right)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\right)\right|\le\left|\frac{R_n(x)}{x}\right|$$

so

$$\begin{aligned}
\left|\int_0^{1/2}\frac{\sin x}{x}dx-\int_0^{1/2}\left(\frac{1}{x}\right)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\right)dx\right| &\le \int_0^{1/2}\left|\frac{R_n(x)}{x}\right|dx \\
&\le \int_0^{1/2}\frac{1}{x}\frac{x^{n+1}}{(n+1)!}dx \\
&\le \int_0^{1/2}\frac{x^n}{(n+1)!}dx \\
&= \left[\frac{x^{n+1}}{(n+1)(n+1)!}\right]_0^{1/2} \\
&= \frac{1}{2^{n+1}(n+1)(n+1)!}
\end{aligned}$$

When $n=6$, this is $\frac{1}{2^7(7)7!}$.

21.1. **Binomial series.** Recall, if $n>0$ that

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$$

where $\binom{n}{k}=\frac{n!}{k!(n-k)!}$.

Definition 6. *For α and real number and k a positive integer, define $\binom{\alpha}{k}$ as follows:*

$$\binom{\alpha}{0}=1,\quad \binom{\alpha}{k}=\frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{k!}$$

Theorem 21 (Binomial Series Theorem)**.** *Let $\alpha \in \mathbb{R}$. Then the McLaurin series for*

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

with interval of convergence $(-1, 1)$.

Example 47. *Find a McLaurin series for* $\frac{1}{\sqrt{9-x}}$.
Solution:

$$\begin{aligned}
\frac{1}{\sqrt{9-x}} &= \frac{1}{3}\left(1+(-x/9)\right)^{-1/2}, \qquad \textit{let } u = x/9 \\
&= \frac{1}{3}\left(1 + \frac{-1/2}{1!}(-u) + \frac{(-1/2)(-1/2-1)}{2!}(-u)^2 + \frac{(-1/2)(-1/2-1)(-1/2-2)}{3!}(-u)^3 + \dots\right) \\
&= \frac{1}{3}\left(1 + \frac{1}{2}u + \frac{1\cdot 3}{2^2 2!}u^2 + \frac{1\cdot 3\cdot 5}{2^3 3!}u^3 + \dots + \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}u^n + \dots\right) \\
&= \frac{1}{3}\sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}u^n
\end{aligned}$$

So, now the McLaurin series for $f(x) = \frac{1}{3}\sum_{n\geq 0} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n! 9^n} x^n$ *we can compute* $f^{(12)}(0)$ *since we know the series! Indeed,*

$$\frac{f^{(12)}(0)}{12!} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{3\cdot 2^{12} 12! 9^n} \Rightarrow f^{(12)}(0) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{3\cdot 2^{12}\cdot 9^n}$$

21.2. **One last application.** The pendulum of length L;

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\phi}}d\phi$$

We would like to approximate T given ϕ_0

Express

$$\frac{1}{\sqrt{1-k^2\sin^2\phi}} = \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}(k^2\sin^2\phi)^n = \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}k^{2n}\sin^{2n}\phi$$

Then

$$4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\phi}}d\phi = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}k^{2n}\sin^{2n}\phi d\phi$$