NOTE TO PRINTER

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THIS EXAMINATION SHOULD BE PRINTED ON $8\frac{1}{2} \times 14$ PAPER, AND STAPLED WITH 3 SIDE STAPLES, SO THAT IT OPENS LIKE A LONG BOOK.

McGILL UNIVERSITY
FACULTY OF SCIENCE
FINAL EXAMINATION

MATHEMATICS 189–340B

ABSTRACT ALGEBRA AND COMPUTING

EXAMINER: Professor W. G. Brown — DATE: Wednesday, April 17th, 2002
ASSOCIATE EXAMINER: Professor W. O. J. Moser — TIME: 09:00 – 12:00 hours

SURNAME: ____________________ SEAT NO.: ___
MR, MISS, MS, MRS, &c.: ____
GIVEN NAMES: ____________________

STUDENT NUMBER: _________ COURSE AND YEAR: _____

Instructions

1. Fill in the above clearly.
2. Do not tear pages from this book; all your writing — even rough work — must be handed in.
3. Calculators are not permitted.
4. This examination booklet consists of this cover, Pages 1 through 7 containing questions; and Pages 8 and 9, which are blank.
5. Show all your work. All solutions are to be written in the space provided on the page where the question is printed. When that space is exhausted, you may write *on the facing page*. Any solution may be continued on the last pages, or the back cover of the booklet, but you must indicate any continuation clearly on the page where the question is printed!
6. You are advised to spend the first few minutes scanning the problems. (Please inform the invigilator if you find that your booklet is defective.)

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1(a) /6	1(b) /8	2 /16	3 /12	4 /12	5(a) /6	5(b) /10	6(a) /6
6(b) /8	7(a) /8	7(b) /8					
				Raw /100	Scaled /100	TERM /30	

1. (a) [6 MARKS] Prove that, for any positive integer n, $3^{2n} + 7$ is a multiple of 8.

   (b) [8 MARKS] Let $\varphi(n)$ be the Euler totient function. Determine all integers $n > 1$ such that $\varphi(n^2) = (\varphi(n))^2$.

2. [16 MARKS] Describe four groups of order 8, no two of which are isomorphic. For all pairs of these groups you are expected to explain precisely how you know they are not isomorphic.

3. [12 MARKS] Carefully showing all your work, determine all integers that satisfy all of the following congruences simultaneously:

$$x \equiv 7 \pmod{5} \tag{1}$$
$$x \equiv -5 \pmod{6} \tag{2}$$
$$x \equiv 3 \pmod{11} \tag{3}$$

Should you require the inverse of an integer which is different from both 1 and -1 to a given modulus n, you are expected to use only the Euclidean algorithm to find it.

4. [12 MARKS] In the ring $\mathbb{Q}[x]$ let $f(x) = x^3 + x + 1$, $g(x) = x^4 + x^3 + 1$. Showing all your work, determine polynomials $a(x)$ and $b(x)$ such that

$$(f(x), g(x)) = f(x) \cdot a(x) + g(x) \cdot b(x)\,,$$

where $(f(x), g(x))$ is the greatest common divisor of $f(x)$ and $g(x)$.

5. (a) [6 MARKS] Prove or disprove: *The polynomial* $x^4 + x^2 + 1$ *is irreducible over* $\mathbb{Z}_{101}$.

   (b) [10 MARKS] Let R_1 denote the ring $\mathbb{R}[x]/(x^2 + 1)$, and let R_2 denote the ring

$$\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| \, a \in \mathbb{R}, b \in \mathbb{R} \right\} ,$$

   (with the usual definitions of matrix addition and multiplication). Prove or disprove: *The mapping* $[a + bx] \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ *is an isomorphism of ring* R_1 *with ring* R_2.

6. (a) [6 MARKS] You are given a nonabelian group G. Consider the subset $H = \{g^{-1} | g \in G\} \subseteq G$. Prove or disprove: H is a subgroup of G.

   (b) [8 MARKS] You are given a ring R, with operations of addition $+_R$ and multiplication $\times_R$, and with additive identity 0_R. Prove or disprove: The set of elements of ring R may be made into another ring S by defining addition $+_S = +_R$, but defining multiplication $\times_S$ by $(\forall a)(\forall b)[a \times_S b = 0_R]$.

7. Let K be a subgroup of a group G (whose operation is written by juxtaposition), and define a binary relation '$\equiv \pmod{K}$' on G by

$$a \equiv b \pmod{K} \quad \Leftrightarrow \quad ab^{-1} \in K\,.$$

   (a) [8 MARKS] Prove that $\equiv \pmod{K}$ is an equivalence relation on G; you are expected to provide justification for every step in your proof.

   (b) [8 MARKS] When $G = S_4$, and $K = \left\langle \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \right\rangle = \langle (3412) \rangle$, determine the equivalence classes of elements of G under the relation. Show your work carefully.

CONTINUATION PAGE FOR PROBLEM NUMBER

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