NOTE TO PRINTER

(These instructions are for the printer. They should not be duplicated.)

THIS EXAMINATION SHOULD BE PRINTED ON $8\frac{1}{2} \times 14$ PAPER, AND STAPLED WITH 3 SIDE STAPLES, SO THAT IT OPENS LIKE A LONG BOOK.

McGILL UNIVERSITY
FACULTY OF SCIENCE
FINAL EXAMINATION

MATHEMATICS 189–340B

ABSTRACT ALGEBRA AND COMPUTING

EXAMINER: Professor W. G. Brown — DATE: Monday, April 30th, 2001
ASSOCIATE EXAMINER: Professor W. O. J. Moser — TIME: 09:00 – 12:00 hours

SURNAME: ____________________ SEAT NO.: ___

MR, MISS, MS, MRS, &c.: ____

GIVEN NAMES: ____________________

STUDENT NUMBER: _________ COURSE AND YEAR: _____

Instructions

1. Fill in the above clearly.
2. Do not tear pages from this book; all your writing — even rough work — must be handed in.
3. Calculators are not permitted.
4. This examination booklet consists of this cover, Pages 1 through 8 containing questions; and Pages 9 and 10, which are blank.
5. Show all your work. All solutions are to be written in the space provided on the page where the question is printed. When that space is exhausted, you may write *on the facing page*. Any solution may be continued on the last pages, or the back cover of the booklet, but you <u>must</u> indicate any continuation clearly on the page where the question is printed!
6. You are advised to spend the first few minutes scanning the problems. (Please inform the invigilator if you find that your booklet is defective.)

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1 /8	2 /8	3 /8	4 /10	5(a) /4	5(b) /6	5(c) /4	6 /6
7 /8	8 /8						
				Raw /70	Scaled /100	TERM /30	

1. [8 MARKS] Prove or disprove:

   It is possible to define a homomorphism from $\mathbb{Z}/7\mathbb{Z}$ *to* $\mathbb{Z}/4\mathbb{Z}$ *by mapping, for all* $a \in \mathbb{Z}$, $[a]_7$ *on to* $[a]_4$.

2. [8 MARKS] Give an explicit example of a ring with at least 8 elements which has no zero divisors, or show that no such ring exists.

3. [8 MARKS] Showing all your work, determine all positive integers n such that $\varphi(n)$ is even (where φ is the Euler *totient* function).

4. [10 MARKS] Carefully showing all your work, determine all integers that satisfy all of the following congruences simlutaneously:

$$x \equiv 1 \pmod{2} \quad (1)$$
$$x \equiv 3 \pmod{5} \quad (2)$$
$$x \equiv 5 \pmod{12} \quad (3)$$
$$x \equiv 8 \pmod{15} \quad (4)$$

(Should you require the inverse of an integer n to a given modulus, you should use a systematic method to find it, unless $n = \pm 1$.)

5. (a) [4 MARKS] Showing all your work, prove carefully that the polynomial $x^4 + x^3 + x^2 + x + 1$ is irreducible over $\mathbb{Z}_2$, but is reducible over $\mathbb{Z}_5$.

   (b) [6 MARKS] Discuss in detail how one can use the polynomial $x^4 + x^3 + x^2 + x + 1$ to construct a field $\mathbb{F}_{16}$ of order $2^4 = 16$. Show the inverses of at least 2 of the elements of the field.

   (c) [4 MARKS] Determine the order of the element x in the multiplicative group of the field that you have constructed.

6. [6 MARKS] Prove or disprove:

There exists a residue class $[a]_8$ *such that,*

$$\forall n \in \mathbb{Z} \left(n \equiv 1 \pmod 2 \Rightarrow n^2 \in [a]_8 \right) .$$

7. [8 MARKS] Recall that the symmetric group S_n consists of all permutations f of the set $\{1, 2, ..., n\}$. Determine, in $\mathcal{S}_4$, all cosets of the subgroup H of all permutations of the symbols 1, 3, 4. (H is the subgroup consisting of the elements $f \in \mathcal{S}_4$ for which $f(2) = 2$.)

8. [8 MARKS] Let a relation $\sim$ on the set $\mathbb{Z} - \{0\}$ be defined as follows:

$$a \sim b \quad \Leftrightarrow \quad \begin{cases} \text{every positive prime } p \text{ which divides } a \text{ also divides } b\,, \\ \text{or} \\ \text{every positive prime } p \text{ which divides } b \text{ also divides } a\,, \\ \text{or both.} \end{cases}$$

Determine carefully whether $\sim$ is an equivalence relation.

CONTINUATION PAGE FOR PROBLEM NUMBER

You *must* refer to this continuation page on the page where the problem is printed!

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