Resurgence in Geometry and Physics

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Lecture 7

Abstract

We describe an application of resurgence to geometry: the classification of singularities of holomorphic foliations.

Contents

For the remainder of the course, we will turn our attention to some applications of resurgence and links with physics. In this lecture, we will focus on an application of resurgence to geometry: the local classification of singular foliations in two dimensions. To appreciate the significance, we begin with a brief review of the classical theory.

1 Classical results on planar foliations

1.1 Planar foliations

The focus of this lecture is on the study of nonlinear ordinary differential equations (ODEs) of the form

$$P(x,y)dx - Q(x,y)dy = 0$$

where P and Q are holomorphic functions. When we say that this equation is an ODE, we are being a bit imprecise, because we have not specified the independent variable. But if Q is nonvanishing in a neighbourhood of the origin $x = y = 0$, we can rewrite the equation in the more familiar form

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}.$$

Then solutions for y as a function of x will exist, at least for (x, y) sufficiently close to the origin. Such a solution is uniquely determined by its initial condition when $x = 0$. Likewise, if $P \neq 0$, we can write

$$\frac{dx}{dy} = \frac{Q(x,y)}{P(x,y)},$$

so we can solve for x as a function of y.

Our aim is to understand the qualitative behaviour of the solutions, rather than the exact quantitative behaviour. We will therefore allow ourselves to make coordinate changes that simplify the ODE. From this point of view, the case in which P or Q is nonvanishing is somewhat boring; it is called the ***nonsingular*** case. When the equation is nonsingular, we can always choose our coordinates in order to make the equation trivial. To see this, let us assume without loss of generality that Q is nonvanishing near the origin, so that solutions are defined in a neighbourhood $\mathsf{U} \subset \mathbb{C}^2$ of $0 \in \mathbb{C}^2$, and each solution intersects the y-axis in a unique point. We then define a map $\mathsf{U} \to \mathbb{C}^2$ by sending the point (x, y) to the point (x, u), where u is the y-intercept of the solution that passes through (x, y); see Figure 1.

Now (x, u) is a new coordinate system in which the ODE becomes the trivial equation

$$\frac{du}{dx} = 0,$$

or in other words

$$du = 0.$$

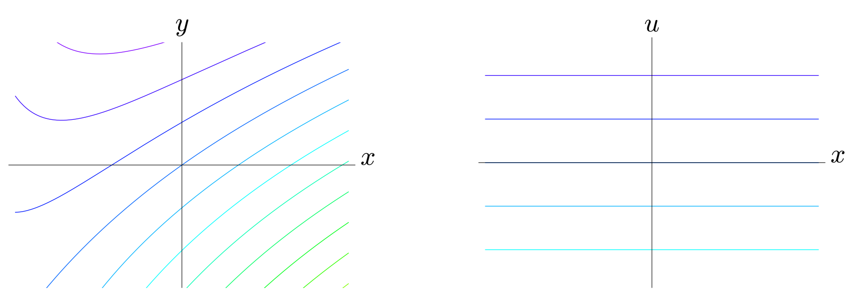

Figure 1: Straightening out the solutions of a nonsingular ODE

So, in these coordinates, the solutions have been "straightened out"; they are simply horizontal lines.

The other thing to notice is that the ODE is unchanged if we multiply P and Q by a nonzero function. In other words, two nonzero one-forms

$$\alpha = P\,dx - Q\,dy \qquad\qquad \widetilde{\alpha} = \widetilde{P}\,dx - \widetilde{Q}\,dy$$

define the same ODE if and only if α and $\widetilde{\alpha}$ are linearly independent on an open dense set, i.e. if and only if

$$\alpha \wedge \widetilde{\alpha} = 0.$$

We formalize this observation in the following geometric definition:

Definition 1. A ***germ of a planar foliation*** is an equivalence class $\mathcal{F} = [\alpha]$ of germs at the origin of nonzero holomorphic one-forms α, where $[\alpha] = [\widetilde{\alpha}]$ if and only if $\alpha \wedge \widetilde{\alpha} = 0$.

From this perspective the solutions of the ODE are simply immersed complex curves $i : \mathsf{Y} \hookrightarrow \mathbb{C}^2$ such that $i^*\alpha = 0$. The are called the ***leaves of the foliation.***

Our aim is to understand the possible local structures of planar foliations, i.e. to classify the germs of planar foliations, up to coordinate transformations. In other words, we would like to describe the structure of the orbit space

$$\frac{\mathsf{Fol}(\mathbb{C}^2, 0)}{\mathsf{Aut}(\mathbb{C}^2, 0)}$$

where $\mathsf{Fol}(\mathbb{C}^2, 0)$ denotes the set of all germs of planar foliations, and $\mathsf{Aut}(\mathbb{C}^2, 0)$ is the set of germs of analytic diffeomorphisms of $\mathbb{C}^2$ that fix the origin. The action of $\phi \in \mathsf{Aut}(\mathbb{C}^2, 0)$ on $\mathcal{F} \in \mathsf{Fol}(\mathbb{C}^2, 0)$ is defined simply by the pullback of forms:

$$\phi^*[\alpha] = [\phi^*\alpha].$$

In light of our considerations above, the classification is trivial if the foliation can be described by a one-form α that is not equal to zero at the origin (so the foliation is nonsingular). When this is not possible, the foliation germ is said to be ***singular***. We now consider the simplest singularities: the linear ones.

1.2 The linear case

Suppose that the foliation germ $\mathcal{F}$ may be defined by a one-form α whose coefficients are homogeneous linear functions:

$$\alpha = (Ax + By)dx + (Cx + Dy)dy$$

where $A, B, C, D \in \mathbb{C}$ are constants. It is useful to consider the matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

defined by the coefficients. (Invariantly, this matrix should be thought of as the bilinear form on the tangent space $\mathsf{T}_0\mathbb{C}^2$ that is obtained by differentiating α at the origin.)

There are two possibilities: either M is diagonalizable, or it is equivalent to a Jordan block. In the diagonalizable case, we can make a linear coordinate change ϕ that brings α into the normal form

$$\alpha_\lambda = x\,dy - \lambda y\,dx$$

where $\lambda \in \mathbb{C}$ is the ratio of the eigenvalues of M. If one of the eigenvalues is zero, we set $\lambda = 0$. Notice that, if $\lambda \neq 0$, then the forms α_λ and $\alpha_{1/\lambda}$ are equivalent by a coordinate change that swaps the roles of x and y. But if λ_1 and λ_2 are nonzero numbers with $\lambda_1 \neq \lambda_2^{\pm 1}$, then the foliations defined by α_{λ_1} and α_{λ_2} cannot be related by any coordinate change; they lie in genuinely different $\mathsf{Aut}(\mathbb{C}^2, 0)$-orbits in $\mathsf{Fol}(\mathbb{C}^2, 0)$.

Exercise 1. Verify these assertions.

The leaves of the foliation defined by α_λ are easily determined. Rearranging the equation $\alpha_\lambda = 0$, we obtain the equation

$$\frac{dy}{y} = \lambda \frac{dx}{x}$$

which means that the solutions are given by $y = Cx^\lambda$ where C is a constant. The only other leaf is given by the y-axis. The case $\lambda = 2$ is illustrated in Figure 2.

Similarly, in the case of a Jordan block, α is equivalent to the normal form

$$x\,dy - (x + y)\,dx,$$

and the leaves are given by the y-axis, and the curves $y = x(\log x + C)$ with $C \in \mathbb{C}$. This case is also illustrated in Figure 2.

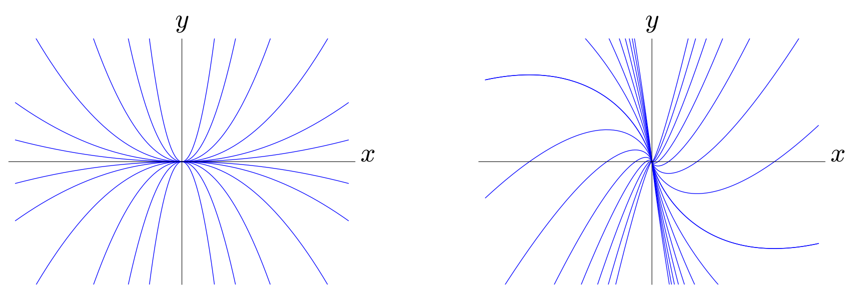

Figure 2: The structure of linear foliations. The left image shows the leaves of the foliation defined by the form $\alpha = ydx\ - 2x\,dy$, while the right image shows the leaves of $x\,dy - (x + y)\,dx$.

1.3 Linearization

Now suppose that we are given a germ of a foliation $\mathcal{F} = [\alpha]$, and we want to find a nice representative for its $\mathsf{Aut}(\mathbb{C}^2, 0)$-orbit. Considering the discussion in the previous section, we can assume that the linear part of α has been put in standard form, corresponding to a diagonal matrix or a Jordan block. For simplicity, let us assume that we are in the diagonal case, so that the Taylor expansion reads

$$\alpha = y\,dx - \lambda x\,dy + \alpha_2 + \alpha_2 + \cdots$$

where α_k for $k > 0$ are one-forms whose coefficients are homogeneous polynomials of degree k.

Let us try to determine when α lies in the same orbit as its linear part. Thus we seek a transformation $\phi \in \mathsf{Aut}(\mathbb{C}^2, 0)$ such that

$$\phi^*\alpha = y\,dx - \lambda x\,dy$$

We will try to find ϕ by induction, constructing a sequence

$$\phi_3, \phi_4, \ldots \in \mathsf{Aut}(\mathbb{C}^2, 0)$$

such that

$$\phi_k^*\alpha = y\,dx - \lambda x\,dy + (\text{terms of order} \geq k)$$

for all $k \geq 3$. Then, by taking the limit

$$\phi = \lim_{k\to\infty} \phi_k,$$

we obtain the desired transformation.

There are two issues that we need to address. First, we need to determine whether it is actually possible to find the transformations ϕ_k for $k \geq 2$. And second, supposing that these transformation do exist, we need to determine whether the limit $\lim_{k\to\infty} \phi_k$ converges.

To deal with the first point, let us assume by induction that the transformation ϕ_{k-1} has been found, so that

$$\phi_k^*\alpha = y\,dx - \lambda x\,dy + A\,dx + B\,dy + (\text{terms of order} > k),$$

where A and B are homogeneous polynomials of degree k. We must find a transformation $\eta \in \mathsf{Aut}(\mathbb{C}^2, 0)$ such that

$$\eta^*(y\,dx - \lambda x\,dy + A\,dx + B\,dy) = y\,dx - \lambda x\,dy + (\text{terms of order} > k). \qquad (1)$$

It therefore seems reasonable to attempt a coordinate change of the form

$$\eta(x, y) = (x + u(x, y), y + v(x, y)),$$

where u and v are homogeneous polynomials of degree k. Then the pullbacks of the elementary coordinate functions and their differentials are given by

$$\begin{aligned}\eta^* x &= x + u \\ \eta^* y &= y + v \\ \eta^* dx &= dx + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \\ \eta^* dy &= dy + \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy.\end{aligned}$$

From these formulae, it is easy to compute that the property (1) will be satisfied if and only if u and v satisfy the equations

$$\begin{aligned}v + y\frac{\partial u}{\partial x} - \lambda x \frac{\partial v}{\partial x} &= -A \\ y\frac{\partial u}{\partial y} - \lambda u - \lambda x \frac{\partial v}{\partial y} &= -B.\end{aligned}$$

We can write this equation abstractly as

$$D_{\lambda,k}\begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} A \\ B \end{pmatrix}$$

where $D_{\lambda,k}$ is a linear differential operator of order one. This operator acts on the finite-dimensional vector space of pairs of homogeneous polynomials of degree k. We observe that this operator is independent of the one-form α; it depends only on the constant λ that determines the linear part. It is possible to show that, if $\lambda \in \mathbb{C}$ is not a rational number (i.e. $\lambda \notin \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$), then $D_{\lambda,k}$ will be invertible for all k. Thus, for any k, it will be possible to find the desired polynomials u and v, which means that the sequence $\phi_2, \phi_3, \ldots$ of transformations will exist.

Then, because each transformation ϕ_{k+1} only differs from the previous transformation ϕ_k by the addition of some polynomials of degree at least k, the limit

$$\phi = \lim_{k\to\infty} \phi_k$$

is a well-defined power series. However, it is not guaranteed to converge. Checking convergence involves bounding the norm of the inverse operator $D_{\lambda,k}^{-1}$, to ensure that the terms u and v that are added at each stage are not too large. When this is done, it turns out that the resulting series will converge for most values of λ. More precisely, we have the Poincaré linearization theorem:

Theorem 1 (Poincaré)**.** *It λ is not real, or if λ is a positive irrational real number, then the procedure above gives an analytic automorphism $\phi \in \mathsf{Aut}(\mathbb{C}^2, 0)$ such that*

$$\phi^*\alpha = x\,dy - \lambda y\,dx.$$

Thus the $\mathsf{Aut}(\mathbb{C}^2, 0)$-orbit of the foliation defined by α is completely determined by the linear part of α.

A complete proof of this result can be found in many places; see, for example, [4, Sections 4 and 5], where the problem is treated using vector fields instead of one-forms.

2 Resurgence in the non-linearizable case

When the conditions of Poincaré's linearization theorem do not hold, the classification problem becomes much more complicated. At the beginning of the 1900s, Dulac obtained several results in this direction. We will focus now on one of the most degenerate cases: the case in which the linearization is diagonalizable, but has zero as an eigenvalue. Thus

$$\alpha = -y\,dx + (\text{terms of order} \geq 2).$$

Dulac showed that either the form α is linearizable, or we can find an analytic transformation ϕ such that

$$\phi^*\alpha = x^{p+1}\,dy - A(x,y)\,dx,$$

where p is some positive integer, and A is a holomorphic function such that $A(0,y) = y$. This is an example of a ***saddle node*** singularity. This normal form is not optimal, since it is is possible to change A to another function of the same type using an analytic transformation. However, there is always a transformation

$$\Phi : (x,y) \mapsto (x, \phi(x,y)),$$

defined by a possibly divergent formal power series ϕ such that

$$\Phi^*\alpha = x^{p+1}\,dy - y(1+\mu x^p)\,dx$$

for some $\mu \in \mathbb{C}$.

The question of when two such foliations are analytically equivalent, rather than formally equivalent, was not solved until the early 1980s, about 80 years after Dulac's work. The solution, found independently by Écalle and Martinet–Ramis [5], is based on Borel summation, and our aim is to illustrate how the classification works in the simplest case, when $p = 0$ and $\mu = 0$. (The other cases yield very similar results, although the exact formulae and notations are slightly more cumbersome.)

We therefore let $\mathcal{F}_0$ be the foliation germ defined by the form

$$\alpha_0 = x^2\,dy - y\,dx,$$

and we consider the subset

$$\mathsf{Fol}_0(\mathbb{C}^2,0) \subset \mathsf{Fol}(\mathbb{C}^2,0)$$

consisting of foliations that differ from α_0 by a (possibly divergent) formal power series automorphism $\phi \in \widehat{\mathsf{Aut}}(\mathbb{C}^2,0)$. We wish to describe the orbit space

$$\frac{\mathsf{Fol}_0(\mathbb{C}^2,0)}{\mathsf{Aut}(\mathbb{C}^2,0)}$$

determined by actual analytic equivalence, rather than just formal equivalence.

In light of Dulac's results, the orbit space can be presented in the following way:

$$\frac{\mathsf{Fol}_0(\mathbb{C}^2,0)}{\mathsf{Aut}(\mathbb{C}^2,0)} \cong \frac{\{\text{germs } \alpha = x^2\,dy - A\,dx,\ \text{formally equivalent to } \alpha_0\}}{\{\text{holomorphic transformations } \Phi(x,y) = (x,\phi(x,y))\}}.$$

In other words, we reduce the problem to the case in which all of the transformations preserve the fibration $(x,y) \mapsto x$. From this point of view, we are trying to classify certain singular Ehresmann connections on the fibration, up to an appropriate group of gauge transformations.

The solution then proceeds according to the following steps.

1. Find the fibre-preserving transformation that transforms the foliation to the normal form
2. Construct analytic transformation in sectors in x-space by Borel summing the formal transformation
3. Understand how the Borel sums jump across singular rays (the Stokes phenomenon)
4. Use the resulting "Stokes data" to determine the analytic equivalence class

We will only sketch the argument here; we refer the reader to [1, Pro II], [2], [5], [6] and [7] for further details and perspectives.

2.1 The formal transformation

Suppose that we are given a one-form in Dulac's form

$$\alpha = x^2 - (y+A)\,dx,$$

and that α is equivalent to the normal form

$$\alpha_0 = x^2\,dy - y\,dx$$

via a formal transformation $\Phi(x,y) = (x,\phi(x,y))$. Let us determine this transformation by expanding

$$\phi(x,y) = \phi_0(x) + \phi_1(x)y + \phi_2(x)y^2 + \cdots$$

and

$$A = a_0(x) + a_1(x)y + a_2(x)y^2 + \cdots .$$

Then the condition $\Phi^*\alpha = \alpha_0$ yields an infinite collection of ODEs

$$\begin{aligned} x^2\frac{d\phi_0}{dx} - \phi_0 &= A(x, \phi_0) \\ x^2\frac{d\phi_1}{dx} &= \beta\phi_1 + \text{terms involving } \phi_0 \\ x^2\frac{d\phi_2}{dx} + \phi_2 &= \beta\phi_2 + \text{terms involving } \phi_0, \phi_1 \\ &\vdots \\ x^2\frac{d\phi_k}{dx} + (k-1)\phi_k &= \beta\phi_k + \text{terms involving } \phi_0, \dots, \phi_{k-1} \\ &\vdots \end{aligned} \tag{2}$$

where

$$\beta = a_1 + 2a_2\phi_0 + 3a_3\phi_0^2 + \cdots .$$

One can show that the first ODE uniquely determines ϕ_0. Meanwhile, the second equation determines ϕ_1 once we impose the initial condition $\phi_1(0) = 1$. This condition amounts to requiring that

$$\phi(0, y) = y + \text{higher order terms}$$

Then the remaining equations uniquely determine the other series ϕ_k, and yield the following

Lemma 1. *There exists a unique formal transformation $\Phi(x, y) = (x, \phi(x, y))$ such that*

$$\Phi^*\alpha = \alpha_0$$

and

$$\phi(0, y) = y.$$

Thus Φ restricts to the identity map on the fibre $x = 0$.

Exercise 2. Verify this lemma.

2.2 Borel summability of the transformation

In order to examine the Borel summability of the series $\phi_0, \phi_1, \phi_2, \ldots \in \mathbb{C}[[x]]$ we set

$$\omega_k = \widehat{\mathscr{B}}(\phi_k)$$

and consider the Borel transforms of the equations (2). For example, the first equation yields

$$(t-1)\omega_0 = \widehat{\mathscr{B}}(a_0) + \widehat{\mathscr{B}}(a_1) * \omega_0 + \widehat{\mathscr{B}}(a_2) * \omega_0 * \omega_0 + \cdots , \tag{3}$$

where we have used the fact that the Borel transform takes the derivative $x^2\partial_x$ to multiplication by t, and products to convolutions.

Since the coefficient function A is holomorphic, the individual series a_k in its expansion are convergent, and hence their Borel transforms $\widehat{\mathscr{B}}(a_k) \in \Omega^1(\mathsf{T})$ are entire forms of exponential type. So they do not introduce any singularities. However, if we divide (3) by $t-1$, we get an equation that suggests that ω_0 may have a singularity at the point $t = 1$ in T. But then, we expect the iterated convolution ω_0^{*k} to have singularities at $t = 1, 2, \ldots, k$. Feeding these singularities back into (3), we expect that ω_0 has singularities when t takes on positive integer values.

Meanwhile, the equations for ω_k with $k > 1$ have the form

$$(t + (k-1))\omega_k = \widehat{\mathscr{B}}(\beta) * \omega_k + \text{terms involving } \omega_0, \ldots, \omega_{k-1}$$

The singularities of ω_k therefore either come from the singularities of $\omega_0, \ldots, \omega_{k-1}$, or from inverting the factor $t + (k-1)$. In this way, our heuristic suggests that the only singularities of ω_{k-1} will be at the points $t = -k+1, -k+2, \ldots$.

The heuristic argument above can be made rigorous by using a series of approximations to determine the forms ω_k:

Proposition 1 (Écalle, Martinet–Ramis)**.** *Let $\Gamma = t^{-1}(\mathbb{Z}) \subset \mathsf{T}$. Then the forms ω_k are Γ-continuable for all $k \geq 0$, and have at most exponential growth at infinity. Hence they are Borel summable along any ray in T that avoids Γ.*

It is now apparent that there are exactly two singular directions in the problem, namely the rays corresponding the positive and negative real axes in the coordinate t. We therefore have two sectorial neighbourhoods $\mathsf{U}_\pm$ of the origin in x-space over which the series may be summed, as shown in Figure 3.

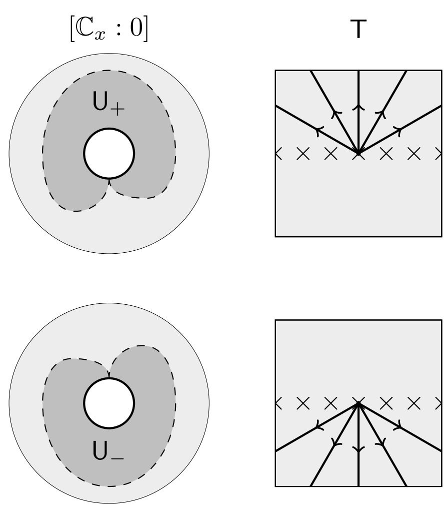

Figure 3: The regions in x-space over which the formal transformation ϕ is Borel summable, and the corresponding rays and singularities in the tangent space T.

We can then construct open neighbourhoods $\widetilde{\mathsf{U}}_\pm$ of $\mathsf{U}_\pm \times \{0\}$ in the real-oriented blowup $[\mathbb{C}_x : 0] \times \mathbb{C}_y$, over which there exist isomorphisms

$$\Phi_\pm(x, y) = (x, \phi_\pm(x, y)) : \mathcal{F}|_{\widetilde{\mathsf{U}}_\pm} \to \mathcal{F}_0|_{\widetilde{\mathsf{U}}_\pm}$$

between our given foliation and the formal normal form. These isomorphisms are Gevrey functions that have the formal transformation $(x, \phi(x, y))$ as their asymptotic expansion with respect to x. In particular, $\Phi_\pm(0, y) = y$.

2.3 The Stokes phenomenon

The Stokes phenomenon for these Borel sums can be understood as follows. Shrinking the open sets $\mathsf{U}_\pm$ if necessary, we may assume that the intersection $\mathsf{U}_+ \cap \mathsf{U}_-$ has two connected components L and R, corresponding to the singular rays on the left and right side of zero in T, as shown in Figure 4. These regions are covered by overlapping regions $\widetilde{\mathsf{L}}, \widetilde{\mathsf{R}} \subset [\mathbb{C}^2 : 0] \times \mathbb{C}$ on which the transformations $\Phi_\pm$ constructed in the previous section are both defined.

On the overlaps, we can form the compositions

$$\Psi_{\mathsf{L}} = \Phi_+|_{\widetilde{\mathsf{L}}} \circ \Phi_-^{-1}|_{\widetilde{\mathsf{L}}} \in \mathsf{Aut}(\mathcal{F}_0|_{\widetilde{\mathsf{L}}})$$
$$\Psi_{\mathsf{R}} = \Phi_+|_{\widetilde{\mathsf{R}}} \circ \Phi_-^{-1}|_{\widetilde{\mathsf{R}}} \in \mathsf{Aut}(\mathcal{F}_0|_{\widetilde{\mathsf{R}}}).$$

These maps are automorphisms of the normal form $\mathcal{F}_0$ that exactly measure the jumps in the Borel sums across the singular rays, and we wish to understand what they look like. Écalle takes a direct computational approach to this problem, using the ODE to produce a "bridge equation" that constrains the action of the alien derivative (see, e.g., [1, Pro II], [2] or [3, Section 6]). We shall instead follow Martinet and Ramis' approach, which is more geometric.

By construction, the automorphisms have the form

$$\Psi_{\mathsf{L}}(x, y) = (x, \psi_{\mathsf{L}}(x, y))$$

and

$$\Psi_{\mathsf{R}}(x, y) = (x, \psi_{\mathsf{R}}(x, y)),$$

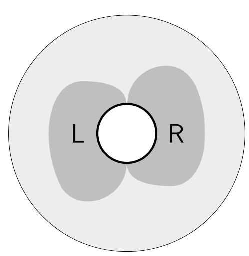

Figure 4: The overlap region $\mathsf{U}_+ \cap \mathsf{U}_- = \mathsf{L} \coprod \mathsf{R}$.

with $\psi_{\mathsf{L}}(0,y)=\psi_{\mathsf{R}}(0,y)=y$. So to understand the possible jumps in the Borel sums, we are left with the problem of determining the automorphisms of the normal form over the L and R that preserve the fibration $(x,y)\mapsto x$.

To this end, we observe that the leaves of the normal form $\mathcal{F}_0$ are easily found. This foliation is defined by the one-form

$$x^2\,dy - y\,dx,$$

and so the solution curves

$$y = Ce^{-1/x}$$

for $C\in\mathbb{C}$ define most of the leaves; the only other leaf is the y-axis. The foliation is illustrated in Figure 5.

Evidently, there is a major qualitative difference between the leaves over the sets L and R; the function $e^{-1/x}$ is decaying in R but blowing up in L. We will use this information to constrain the behaviour of our automorphisms.

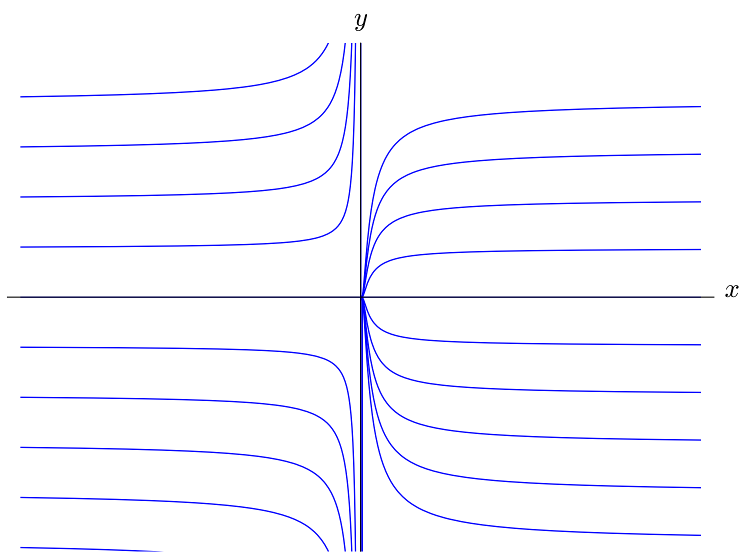

Figure 5: The leaves of the normal form foliation in the real plane $\mathbb{R}^2\subset\mathbb{C}^2$.

First, we notice that any automorphism of $\mathcal{F}_0$ must send leaves to leaves. Since our automorphisms fix x, we must therefore have

$$\psi_{\mathsf{L}}(x, Ce^{-1/x}) = g_{\mathsf{L}}(C)e^{-1/x}$$

and

$$\psi_{\mathsf{R}}(x, Ce^{-1/x}) = g_{\mathsf{R}}(C)e^{-1/x}$$

for some holomorphic functions g_{L} and g_{R}. Using the equation

$$C = ye^{1/x},$$

we arrive at the formulae

$$\psi_{\mathsf{L}}(x,y) = g_{\mathsf{L}}(ye^{1/x})e^{-1/x} \qquad\qquad \psi_{\mathsf{R}}(x,y) = g_{\mathsf{R}}(ye^{1/x})e^{-1/x}.$$

We now analyze the left and right cases separately. Expanding g_{L} in power series, we find

$$\begin{aligned}\psi_{\mathsf{L}}(x,y) &= (g_0 + g_1\, ye^{1/x} + g_2\, y^2e^{2/x} + \cdots)e^{-1/x} \\ &= g_0e^{-1/x} + g_1\, y + g_2\, y^2e^{1/x} + \cdots\end{aligned}$$

where g_k is the kth Taylor coefficient of g_{L}. In the region L, the exponential $e^{1/x}$ decays as $x \to 0$, while $e^{-1/x}$ blows up. Thus, in order for ψ_{L} to have the desired property $\psi_{\mathsf{L}}(0,y) = y$, we must have

$$g_0 = 0$$

and

$$g_1 = 1,$$

but the rest of the coefficients are unconstrained. We conclude that

$$g_{\mathsf{L}}(C) = C + g_2C^2 + g_3C^3 + \cdots$$

defines an automorphism of the germ $(\mathbb{C}, 0)$ that is tangent to the identity map at the special point $C = 0$.

On the other hand, a similar argument in the region R evidently gives the opposite result, since the role of the two exponentials is revered. Therefore the only nonzero Taylor coefficients of g_{R} are the coefficient of C, which is equal to one, and the constant term, which is arbitrary. We conclude that g_{R} defines a translation

$$g_{\mathsf{R}}(C) = C + \tau$$

for some $\tau \in \mathbb{C}$.

Thus, starting from the one-form α, we have constructed two pieces of data from the formal series Φ via the Stokes phenomenon: the number $\tau \in \mathbb{C}$ and the automorphism g_{L} of $(\mathbb{C}, 0)$ that is tangent to the identity. We call these data the ***Stokes data***, and we will use them to complete the classification.

2.4 The classification theorem

Let $\mathsf{Aut}_0(\mathbb{C}, 0) \subset \mathsf{Aut}(\mathbb{C}, 0)$ denote the group of automorphisms of the germ $(\mathbb{C}, 0)$ that are tangent to the identity. From our considerations in the previous section, we see that there is a canonical ***classifying map***

$$\{\text{germs } \alpha = x^2\, dy - A\, dx, \text{ formally equivalent to } \alpha_0\} \to \mathbb{C} \times \mathsf{Aut}_0(\mathbb{C}, 0),$$

defined by extracting the Stokes data (τ, g_{L}) of the one-form α. We may now state the analytic classification theorem:

Theorem 2 (Écalle, Martinet–Ramis)**.** *If α_1 and α_2 are holomorphically equivalent, then their images under the classifying map are equal. The resulting map on the quotient*

$$\frac{\mathsf{Fol}_0(\mathbb{C}^2,0)}{\mathsf{Aut}(\mathbb{C}^2,0)} \to \mathbb{C} \times \mathsf{Aut}_0(\mathbb{C},0)$$

is a bijection.

Sketch of proof. We refer to [5] for the full proof, which requires that some analytic details be checked. We describe here only the main geometric ideas.

If two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ in $\mathsf{Fol}_0(\mathbb{C}^2,0)$ are related by a holomorphic transformation, we will have a commutative diagram

$$\begin{tikzcd} \mathcal{F}_1 \arrow[rr, "\rho"] \arrow[dr, "\Phi_1"'] & & \mathcal{F}_2 \arrow[dl, "\Phi_2"] \\ & \mathcal{F}_0 & \end{tikzcd}$$

relating their formal isomorphisms with the normal form $\mathcal{F}_0$.

Because ρ is holomorphic, it does not change as we cross the singular rays, and hence we have a similar identity relating the Borel sums of the formal series, namely

$$\Phi_{1,\pm} = \Phi_{2,\pm} \circ \rho$$

on the overlap of their domains. This immediately implies that the Stokes data for these two series are the same:

$$\begin{aligned} \Psi_{1,\mathsf{L}} &= \Phi_{1,+} \circ \Phi_{1,-}^{-1} \\ &= \Phi_{2,+} \circ \rho \circ (\Phi_{2,-} \circ \rho)^{-1} \\ &= \Phi_{2,+} \circ \rho \circ \rho^{-1} \circ \Phi_{2,-}^{-1} \\ &= \Phi_{2,+} \circ \Phi_{2,-}^{-1} \\ &= \Psi_{2,\mathsf{L}} \end{aligned}$$

on the appropriate overlap. Similarly, $\Psi_{1,\mathsf{R}} = \Psi_{2,\mathsf{R}}$. Hence the two foliation have the same image under the classifying map, i.e. the classifying map descends to the quotient, as claimed.

To see that the map is injective, we essentially reverse the previous calculation. Indeed, suppose that Φ_1 and Φ_2 have the same Stokes data. Then over appropriate regions $\widetilde{\mathsf{U}}_\pm$, their Borel sums give isomorphisms

$$\rho_\pm = \Phi_{2,\pm}^{-1} \circ \Phi_{1,\pm}$$

between $\mathcal{F}_1$ and $\mathcal{F}_2$. A calculation nearly identical to the one above shows that ρ_+ and ρ_- agree on the overlap of their domains, and hence they can be glued together to give a single holomorphic isomorphism

$$\rho : \mathcal{F}_1 \to \mathcal{F}_2.$$

defined in a neighbourhood of the origin in $\mathbb{C}^2$. Thus $\mathcal{F}_1$ and $\mathcal{F}_2$ lie in the same orbit.

Finally, the surjectivity is established as follows. Starting with the Stokes data (τ, g_{L}), we can construct the corresponding automorphisms Ψ_{L} and Ψ_{R} of the normal form $\mathcal{F}_0$ over the regions L and R. We then use these automorphism to glue $\mathcal{F}_0|_{\tilde{\mathsf{U}}_+}$ to $\mathcal{F}_0|_{\tilde{\mathsf{U}}_-}$ on the appropriate overlap $\widetilde{\mathsf{L}} \coprod \widetilde{\mathsf{R}}$. This results in a foliation that is formally equivalent to $\mathcal{F}_0$, but has the prescribed Stokes data. $\square$

Remark 1. The reader familiar with sheaf cohomology may recognize that the Stokes data define a Čech 1-cocycle for the sheaf of automorphisms of the formal normal form; this is the viewpoint taken in Martinet and Ramis' paper [5].

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