Resurgence in Geometry and Physics

Brent Pym

Trinity Term 2016
Lecture 6

Abstract

We introduce the Stokes automorphism, which controls the jump in a Borel sum as we cross a singular ray. It has a natural logarithm, called the alien derivative. Using complex powers of the Stokes automorphism, we introduce a whole family of summation operators that interpolate between the two lateral summations. This allows us to assign unambiguous real-valued sums to power series with real coefficients.

Contents

1 Motivation: real-valued summations

In the previous lecture, we introduced the Borel summation procedure, which takes a (summable) resurgent series f and produces a function. This procedure depends on a choice of tangent direction $\alpha \in \mathsf{S}_p\mathsf{X}$ in our Riemann surface. As long as the ray is nonsingular for f, the sum $s_\alpha f$ gives a well-defined germ of a function, defined in a sectorial neighbourhood with an opening angle of π.

But when α is a singular direction, there were two different sums $s_{\alpha_\pm}$ which correspond to taking the Laplace transform along contours in T which pass just to the left or right of α. This led to an ambiguity in the summation: $s_{\alpha_+} \neq s_{\alpha_-}$.

When the Borel transform had simple singularities, we found that the ambiguity could be computed in terms of further Borel sums:

$$(s_{\alpha_-} - s_{\alpha_+})f = s_{\alpha_+}\left(\sum_{k=1}^{\infty} f_k e^{-a_k/x}\right)$$

where the coefficients a_k are precisely the coordinates of the singularities of the Borel transform in T, and the coefficient series f_k are dictated by the structure of the singularities (the minor and the residue); we will be more precise below.

In many applications, we will be trying to obtain a real-valued function, such as the solution of a differential equation on the real-line, or the expectation value of some observable in a quantum field theory. Hence the series f in question will have real coefficients, and we will want to obtain a real-valued sum by taking our Borel sum along the positive real axis. But in many interesting cases, the real axis will turn out to be a singular direction for the problem, and the resulting sums will be not only ambiguous, but also complex-valued.

For example, consider the series $f = -\sum_{k=0}^{\infty} k!(x/a)^{k+1}$ with $a > 0$. Then the Borel transform is given by

$$\omega = \widehat{\mathscr{B}}(f) = \frac{dt}{t-a} \in \Omega^1(\mathsf{T}_\Gamma).$$

In this case, the positive real axis is a singular direction. The left and right Borel sums can be easily computed by integrating along contours $\alpha_\pm$ that stick to the positive real axis except for small semi-circles that avoid the singularity at $t = a$. By sending the radius of the semi-circle to zero, we easily get the following expression:

$$s_{\alpha_\pm} f = \int_{\alpha_\pm} \frac{e^{-t/x}dt}{t-a} = \text{P.V.} \int_0^\infty \frac{e^{-t/x}dt}{t-a} \mp \pi i e^{-a/x}$$

where the real part

$$f_0 = \text{Re}(s_{\alpha_\pm}) = \text{P.V.} \int_0^\infty \frac{e^{-t/x}dt}{t-a}$$

is independent of the contour, and is given by the Cauchy principal value of the integral.

This happens more generally: when we apply the lateral summations to a real series, we will always get answers that are complex conjugates of one another. This suggests one way to cure the problem: whenever we try to sum up a real-valued series, we should simply take the real part $\text{Re}(s_{\alpha_+}) = \text{Re}(s_{\alpha_-})$. The result is both unambiguous and real, which seems to solve both of our problems at once.

But this simple solution fails a crucial test: it behaves poorly with respect to multiplication. To see this, we note that since $s_{\alpha_\pm}$ are built directly from the

Borel and Laplace transforms, they are algebra homomorphisms. Thus, in our example, we have

$$\begin{aligned} s_{\alpha_\pm}(f^2) &= (s_{\alpha_\pm}(f))^2 \\ &= (f_0 \mp \pi i e^{-a/x}) \\ &= f_0^2 \mp 2\pi i e^{a/x} - \pi^2 e^{-2a/x}, \end{aligned}$$

and taking the real part, we find

$$\mathrm{Re}(s_{\alpha_\pm} f^2) = f_0^2 - \pi^2 e^{-2a/x}.$$

This result is evidently different from

$$\mathrm{Re}(s_{\alpha_\pm} f)^2 = f_0^2.$$

Thus the naive operation of taking the real part $\mathrm{Re}(s_{\alpha_\pm})$ is *not* an algebra homomorphism.

Our aim in this lecture is to discuss Écalle's solution to the problem via "alien calculus". The basic idea is to produce a summation procedure that is somehow halfway in between the two lateral sums, but it more clever than just taking an average.

2 Stokes automorphisms and alien calculus

2.1 The Stokes automorphism and median summation

In Lecture 5, we introduced the algebra of resurgent symbols $\mathcal{R}(\mathsf{A})$ in a sector $\mathsf{A} \subset \mathsf{S}_p\mathsf{X}$. Elements of this algebra are formal expressions of the form

$$f = f_0 + \sum_{0 \neq v \in \Gamma} f_v e^{-t(v)/x} = \sum_{v \in \Gamma} f_v e^{-t(v)/x}$$

where the coefficients $f_v \in \widehat{\mathcal{O}}_{\mathsf{X},p} \cong \mathbb{C}[[x]]$ are simple resurgent functions, and the sum is taken over a discrete subset $\Gamma \subset \mathsf{T}$, chosen so that the exponentials $e^{-t(v)/x}$ for $v \in \Gamma \setminus \{0\}$ decay in the sector A. We denote by $\mathcal{R}_0 \subset \mathcal{R}(\mathsf{A})$ the subalgebra consisting of resurgent symbols that have no exponential terms, i.e. for which $f = f_0 \in \mathbb{C}[[x]]$.

We can extend the Borel sum for formal power series to all resurgent symbols by the expression

$$s_{\alpha_\pm} f = \sum_{v \in \Gamma} (s_{\alpha_\pm} f_v) e^{-t(v)/x},$$

assuming that all of the Borel sums exist and the resulting series of functions converges.

The relation between left and right sums of a resurgent symbol is rather complicated, since there are contributions from the ambiguities in the Borel sums for all of the different series f_v. We will now set some notational conventions for keeping track of these contributions.

First, we observe that every vector $v \in \mathsf{T} \setminus \{0\}$ determines an operator S_v, which acts on the algebra $\mathcal{R}_0$ of simple resurgent functions; the action $S_v \cdot f$ on a simple resurgent function $f \in \mathcal{R}_0$ is determined as follows. First, take the Borel transform $\omega = \widehat{\mathscr{B}}(f)$. By assumption, ω has an endless analytic continuation away from some discrete set $\mathsf{sing}(\omega) \subset \mathsf{T}$ of singularities. Denote by γ the homotopy class of a path from the origin to v that is obtained by following the ray from 0 to v, but making a small detour to the right of all of the points of $\mathsf{sing}(\omega)$. We can then extract the residue $\mathrm{Res}_\gamma \omega$ and the minor ω_γ, as described in Section 1.4 of Lecture 5. Using these data, we define the action

$$S_v \cdot f = 2\pi i \mathrm{Res}_\gamma \omega + \widehat{\mathscr{L}}(\omega_\gamma) \in \mathcal{R}_0 \subset \mathbb{C}[[x]]$$

where $\widehat{\mathscr{L}} = \widehat{\mathscr{B}}^{-1}$ is the formal Laplace transform. Notice that

$$S_v \cdot f = 0$$

if ω has no singularity at v. In particular, if f is a convergent series, then ω will be entire, and hence $S_v f = 0$ for all v.

Now let

$$f = \sum_{v \in \Gamma} f_v e^{-t(v)/x}$$

be a resurgent symbol. It follows immediately from Proposition 3 in Lecture 5 that the jump in the Borel sum of f along a ray α in T may be computed as follows:

$$\begin{aligned} s_{\alpha_-} f - s_{\alpha_+} f &= \sum_{v \in \Gamma} (s_{\alpha_-} f_v - s_{\alpha_+} f_v) \cdot e^{-t(v)/x} \\ &= s_{\alpha_+} \sum_{v \in \Gamma} \sum_{w \in \alpha} S_w(f_v) e^{-t(v+w)/x}. \end{aligned}$$

This equation has many terms, but we notice that it has the following basic structure:

$$s_{\alpha_-} = s_{\alpha_+} \circ (1 + \delta_\alpha)$$

where

$$\delta_\alpha = \sum_{w \in \alpha} e^{-t(w)/x} S_w : \mathcal{R}(\mathsf{A}) \to \mathcal{R}(\mathsf{A}) \tag{1}$$

is the operator that extracts the formal contributions from all the singularities along the ray α. The key point about δ_α is that it only ever adds exponentially small corrections, so it is "small" in an appropriate sense. We have

$$(1 + \delta_\alpha) f = f + (\text{higher order terms}),$$

so that we hope to be able to recover f from $(1+\delta_\alpha)f$. Indeed, this is the case; as we shall see, the linear operator

$$\mathfrak{S}_\alpha = 1 + \delta_\alpha \in \mathsf{End}(\mathcal{R}(\mathsf{A}))$$

is, in fact, an algebra automorphism. It is called the ***Stokes automorphism*** or the ***crossing automorphism***. In fact, we will see that an operator $\mathfrak{S}_\alpha^\nu$ can be defined for any complex exponent ν, not just $\nu = -1$.

2.2 The alien derivative

In order to construct the powers $\mathfrak{S}_\alpha^\nu$ for $\nu \in \mathbb{C}$, we will use the formula

$$\mathfrak{S}_\alpha^\nu = \exp(\nu \log \mathfrak{S}_\alpha),$$

which means that we need to define a logarithm for $\mathfrak{S}_\alpha$. But we have written

$$\mathfrak{S}_\alpha = 1 + \delta_\alpha,$$

and so the obvious thing to do is to apply the Taylor expansion for the logarithm:

$$\log(1+\delta_\alpha) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\delta_\alpha^n}{n}$$

Equivalently, we could use the Newton binomial series

$$(1+\delta_\alpha)^\nu = \sum_{n=0}^{\infty} \frac{\nu(\nu-1)\cdots(\nu-n+1)}{n!}\delta_\alpha^n.$$

Fortunately, there is no problem with the convergence of these series. The reason is that, for a fixed resurgent symbol $f \in \mathcal{R}(\mathsf{A})$, and a fixed $w \in \mathsf{T}$, the coefficient of $e^{-t(w)/x}$ in $\delta_\alpha^n f$ will vanish for n sufficiently large. This happens because the points of Γ and the singular sets of the coefficients f_v are discrete. As we take higher and higher powers of δ_α, the locations of the singularities add together, moving further and further from the origin.

Exercise 1. Verify this claim.

Remark 1. The convergence of the sum can also be formalized by introducing a convenient topology on the algebra $\mathcal{R}(\mathsf{A})$, designed so that $e^{-t(w)/x}$ is small when $w \in \mathsf{T}$ is large. This topology is not analytic in nature (i.e. there are no norms or estimates involved); it is a purely algebraic device for keeping track of the order of exponentials, similar to the so-called adic topology on the ring of formal power series, or the topology on the Novikov ring used in symplectic topology. Details can be found, for example in [2, Section 2.4.2]. □

From the fact that $\mathfrak{S}_\alpha = 1+\delta_\alpha$ is an algebra automorphism, it follows easily that the logarithm

$$\Delta_\alpha = \log(\mathfrak{S}_\alpha) = \log(1+\delta_\alpha)$$

is a derivation, i.e. it satisfies the Leibniz rule

$$\Delta_\alpha(fg) = \Delta_\alpha(f)g + f\Delta_\alpha(g).$$

Thus Δ_α acts on $\mathcal{R}(\mathsf{A})$ like a derivative. But notice that it acts trivially on any convergent series $f \in \mathcal{O}_{\mathsf{X},p} = \mathbb{C}\{x\} \subset \mathbb{C}[[x]]$ because the Borel transform of such a series has no singularities. This means that we cannot write the operator Δ_α as a usual derivative operator $u(x)\partial_x$. For this reason, Écalle calls Δ_α the ***alien derivative in the direction*** α.

2.3 Alien derivatives of the Euler series

To get a feeling for how alien derivatives works, let us compute some derivatives explicitly in the case of the Euler series

$$f = -\sum_{k=0}^{\infty} k!(x/a)^{k+1},$$

with $a > 0$, whose Borel transform is

$$\omega = \frac{dt}{t-a}.$$

The singularity operator S_w for $w \in \mathsf{T}$ acts on f by extracting the residue:

$$S_w f = \begin{cases} 2\pi i & t(w) = a \\ 0 & \text{otherwise} \end{cases}$$

Thus when α is the ray defining the positive real axis in the coordinate t, we have

$$\delta_\alpha f = 2\pi i\, e^{-a/x}.$$

The coefficient of the exponential is the constant $2\pi i$, which is holomorphic, and hence we have $\delta_\alpha^n f = 0$ for $n > 1$. This gives the alien derivative

$$\Delta_\alpha f = \delta_\alpha f - \tfrac{1}{2}\delta_\alpha^2 f + \tfrac{1}{3}\delta_\alpha^3 f - \cdots = 2\pi i\, e^{-a/x},$$

while all higher derivatives vanish.

Now we can compute the derivatives of arbitrary powers of f using the derivation property:

$$\Delta_\alpha f^k = k f^{k-1}\Delta_\alpha f = 2\pi i k f^{k-1} e^{-a/x}.$$

But it is also instructive to compute the alien derivative of f^2 directly. As we saw in Lecture 4, its Borel transform is given by

$$\widehat{\mathscr{B}}(f^2) = \omega * \omega = \frac{2\log(1-t/a)\,dt}{t-2a}.$$

There are now two singularities, at the points $w, 2w \in \mathsf{T}$ where $t = a$ and $t = 2a$. In order the compute the contribution from the point w, we must compute the minor $(\omega * \omega)_w$, which means that we must write

$$\omega * \omega = g(t-a)\log(t-a)\,dt$$

and extract the coefficient $2\pi i g(t)\,dt$ which measures the branching of $\omega * \omega$ at w. We evidently have

$$(\omega * \omega)_w = \frac{4\pi i\,dt}{t-a},$$

which has formal Laplace transform given by

$$S_w \cdot f^2 = 4\pi i f$$

Meanwhile, the singularity at $2w$ contributes the residue

$$S_w f^2 = 2\pi i \cdot \mathrm{Res}_{2w}(\omega * \omega) = 2\pi i \cdot 2\log(-1) = -4\pi^2$$

Notice that, in order to get the correct sign for $\log(-1)$, we must make sure to stay on the correct sheet of the Riemann surface of $\log(1 - t/a)$, passing just to the right of the singularity at w as we analytically continue to the point $2w$.

Putting these calculations together with the appropriate exponential factors, we obtain

$$\begin{aligned}\delta_\alpha(f^2) &= S_w(f^2)e^{-a/x} + S_{2w}(f^2)e^{-2a/x} \\ &= 4\pi i f e^{-a/x} - 4\pi^2 e^{-2a/x}.\end{aligned}$$

Now δ_α^2 acts nontrivially, giving

$$\delta_\alpha^2(f^2) = 4\pi i \delta_\alpha(f)e^{-a/x} = -8\pi^2 e^{-2a/x}$$

Hence the first alien derivative is given by

$$\begin{aligned}\Delta_\alpha(f^2) &= \delta_\alpha(f^2) - \tfrac{1}{2}\delta_\alpha^2(f^2) + \cdots \\ &= \left(4\pi i f e^{-a/x} - 4\pi^2 e^{-2a/x}\right) - \tfrac{1}{2}(-8\pi^2 e^{-a/x}) \\ &= 4\pi i f e^{-a/x} \\ &= 2f\Delta_\alpha f,\end{aligned}$$

as expected. Thus we see how the contributions of the different singularities arising from the convolution product are precisely cancelled, in order to make the Leibniz rule work.

2.4 The median summation

Now that we have defined the powers $\mathfrak{S}_\alpha^\nu$ of the Stokes automorphism, we may define not just the left and right Borel sums, but in fact a whole family of

summation operators, where we first apply a power of the Stokes operator to change the resurgent symbol before we sum. We thus obtain the summation operators
$$s^\nu_\alpha = s_{\alpha_+} \circ \mathfrak{S}^\nu_\alpha$$
depending on the complex parameter $\nu \in \mathbb{C}$. Since any power of $\mathfrak{S}_\alpha$ is an algebra automorphism, these operators are compatible with products:

$$s^\nu_\alpha(fg) = s^\nu_\alpha(f)s^\nu_\alpha(g)$$

as functions in appropriate sectorial neighbourhoods. This family of operators interpolates between the left and right Borel sums: we have $s^0_\alpha = s_{\alpha_+}$, while $s^1_\alpha = s_{\alpha_+} \circ \mathfrak{S}_\alpha = s_{\alpha_-}$ by definition of the Stokes operator.

Applied to the Euler series, we find using our calculations in the previous section that

$$\begin{aligned}
\mathfrak{S}^\nu_\alpha f &= \exp(\nu\Delta_\alpha)f \\
&= f + (\nu\Delta_\alpha)f + \tfrac{1}{2}\nu\Delta^2_\alpha f + \cdots \\
&= f + \nu \cdot 2\pi i e^{-a/x}
\end{aligned}$$

for all $\nu \in \mathbb{C}$. Therefore

$$\begin{aligned}
s^\nu_\alpha f &= s_{\alpha_+}(f + \nu \cdot 2\pi i e^{-a/x}) \\
&= (s_{\alpha_+} f) + \nu \cdot 2\pi i e^{-a/x} \\
&= \left(\text{P.V.} \int_0^\infty \frac{e^{-t/x}dt}{t-a} - \pi i e^{-a/x}\right) + \nu \cdot 2\pi i e^{-a/x} \\
&= f_0 + \left(\nu - \tfrac{1}{2}\right) 2\pi i e^{-a/x}
\end{aligned}$$

where f_0 denotes the Cauchy principal value integral. We discover two interesting facts:

1. By varying the parameter ν, we are able to obtain all possible solutions of the ODE
$$x^2\partial_x = af + x,$$
of which f is the unique formal solution. (We studied the case $a = -1$ in Lecture 1.)

2. When $\nu = -\frac{1}{2}$, the sum of the series is real-valued:
$$s^{1/2}_\alpha f = \text{P.V.} \int_0^\infty \frac{e^{-t/x}dt}{t-a}$$

These observations are not accidents. Indeed, the Stokes automorphism $\mathfrak{S}_\alpha$ is compatible not just with products, but also with the derivative operator, thanks for the compatibility between differentiation and Borel transforms. What this means is that if f satisfies some differential equations—linear or nonlinear—then its Borel sums $s_\alpha^\nu f$ must satisfy the same differential equation.

The second observation is an instance of the following general result about the ***median summation operator***

$$s_\alpha^{\mathrm{med}} = s_{\alpha_+} \circ \mathfrak{S}_\alpha^{1/2} = s_{\alpha_-} \circ \mathfrak{S}_\alpha^{-1/2},$$

which lies halfway between the left and right sums:

Theorem 1 (Écalle)**.** *The median summation operator assigns real-valued sums to real-valued series, provided that the summation converges.*

For a sketch of the proof, see [1, Section 7].

References

[1] D. Dorigoni, *An Introduction to Resurgence, Trans-Series and Alien Calculus*, 1411.3585.

[2] B. Y. Sternin and V. E. Shatalov, *Borel-Laplace transform and asymptotic theory*, CRC Press, Boca Raton, FL, 1996. Introduction to resurgent analysis.