# Resurgence in Geometry and Physics

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Lecture 5

**Abstract**

We study the Laplace transforms of endlessly continuable one-forms along rays. We focus, in particular, on the behaviour of the transform as the ray sweeps through a direction in which our one-form has singularities. This behaviour motivates the introduction of a class of "simple singularities", which are easy to manage. Passing back over to functions, we briefly introduce Borel sums, the Stokes phenomenon, and resurgent symbols, which are power series that have been augmented by exponentially small corrections.

## Contents



In the previous lecture, we introduced the notion of a resurgent formal power series $f \in \mathbb{C}[[x]]$. "Resurgent" means that the one-form $\omega = \widehat{\mathscr{B}}(f) \in \widehat{\Omega}^1(\mathsf{T})$, the Borel transform of $f$, converges in a disk near the origin, and has an endless analytic continuation to a Riemann surface $\mathsf{T}_\Gamma$ that covers all $\mathsf{T}$, except possibly some discrete subset $\Gamma \subset \mathsf{T}$. Our aim in this lecture is to introduce the operations that "resum" such series to functions in sectorial neighbourhoods of $p$. The strategy is to construct a function asymptotic to $f$ in some sector by taking an appropriate Laplace transform of $\omega$.

# 1 Laplace transforms of endlessly continuable forms

## 1.1 Laplace transforms in nonsingular directions

Throughout this section, we fix a discrete subset $\Gamma \subset \mathsf{T}$ containing the origin, and a $\Gamma$-continuable form $\omega \in \Omega^1(\mathsf{T}_\Gamma)$. We assume that $\omega$ is of exponential type at infinity, meaning that it has at most exponential growth along any path from $0$ to $\infty$ in $\mathsf{T}_\Gamma$.

Let $\alpha$ be a ray in $\mathsf{T}$, i.e. a straight line from the origin to $\infty$ in some fixed direction. If $\alpha$ does not intersect $\Gamma \setminus \{0\}$, we say that $\alpha$ is a ***nonsingular direction***. In this case, we may define the Laplace transform along $\alpha$ by the formula

$$\mathscr{L}_\alpha = \int_\alpha e^{-t/x}\omega$$

will converge provided that the factor $e^{-t/x}$ decays sufficiently rapidly at infinity.

In particular, for the integral to converge for a fixed value of $x$, we need $\mathrm{Re}(t/x) > 0$ along the ray $\alpha$. Thus the difference in phase between $x$ and $t$ is less than $\pi/2$; this defines a sector of directions in $x$ whose opening angle is $\pi$.

We can think of this geometrically as follows. Recalling that $\mathsf{T} = \mathsf{T}_p\mathsf{X}$ is the tangent space of our Riemann surface at the point of interest, we may interpret $\alpha$ as defining a direction in $\mathsf{X}$ at $p$, i.e. a point on the boundary of the real oriented blowup. Then the ***copolar of*** $\alpha$ is the sector $\mathsf{copol}(\alpha) \subset \mathsf{S}_p\mathsf{X}$ that is centred at $\alpha$ and has opening angle $|\mathsf{copol}(\alpha)| = \pi$, as shown in Figure 1. With this notation we have the

**Lemma 1.** *The Laplace transforms $\mathscr{L}_\alpha(\omega)$ defines a function in a sectorial neighbourhood of $p \in \mathsf{X}$ with opening* $\mathsf{copol}(\alpha)$.

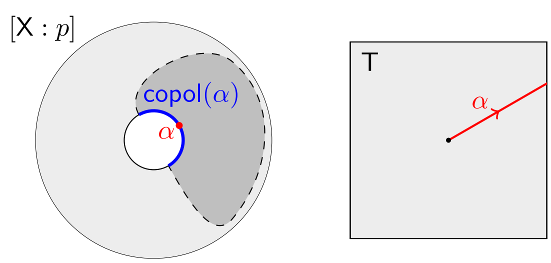


Figure 1: The blue sector denotes the copolar of a ray $\alpha$ in $\mathsf{T}$. The dark grey is a hypothetical region in which the Laplace transform along $\alpha$ would converge.

More generally, if $\mathsf{A} = (\alpha_1, \alpha_2) \subset \mathsf{S}_p\mathsf{X}$ is a sector of angle $|\mathsf{A}| < \pi$, we may associate two other sectors to $\mathsf{A}$. They are the ***polar***

$$\mathsf{pol}(\mathsf{A}) = \bigcap_{\alpha \in \mathsf{A}} \mathsf{copol}(\alpha) = (\alpha_2 - \tfrac{\pi}{2}, \alpha_1 + \tfrac{\pi}{2})$$

and the ***copolar***

$$\mathsf{copol}(\mathsf{A}) = \bigcup_{\alpha \in \mathsf{A}} \mathsf{copol}(\alpha) = (\alpha_1 - \tfrac{\pi}{2}, \alpha_2 + \tfrac{\pi}{2})$$

as shown in Figure 2. Thus, informally, the polar of $\mathsf{A}$ is the region in $\mathsf{X}$ such that $\mathrm{Re}(t/x) > 0$ along *any* ray in $\mathsf{A}$, while the copolar is the region in which there exists *some* ray in $\mathsf{A}$ along which $\mathrm{Re}(t/x) > 0$.

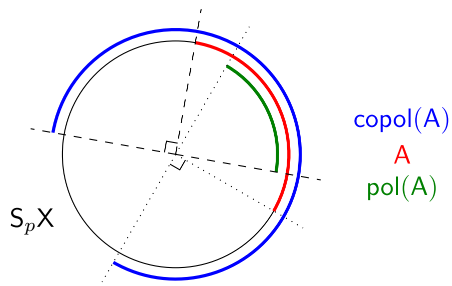


Figure 2: The copolar and copolar of a sector.

Suppose now that we vary the ray $\alpha$ to some other ray $\alpha'$. Then we have two different Laplace transforms $f = \mathscr{L}_\alpha$ and $f' = \mathscr{L}_{\alpha'}$ defined in different regions $\mathsf{U}$ and $\mathsf{U}'$, as shown in Figure 3. We would like to compare these two functions in the overlap $\mathsf{U} \cap \mathsf{U}'$.

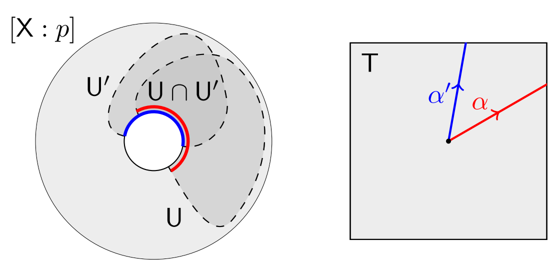


Figure 3: Overlapping domains for a pair of Laplace transforms.

In this region $\mathsf{U} \cap \mathsf{U}'$, we evidently have

$$\mathscr{L}_\alpha(\omega) - \mathscr{L}_{\alpha'}(\omega) = \int_{\alpha - \alpha'} e^{-t/x}\omega.$$

The integrand decays very rapidly at infinity, so provided that the contour $\alpha - \alpha'$ does not encircle any singular points of $\omega$, this integral will be zero by Cauchy's theorem. This gives the

**Proposition 1.** *If the contour $\alpha - \alpha'$ encircles no singular points of $\omega$, then the corresponding Laplace transforms are equal on the intersection of their domains:*

$$\mathscr{L}_\alpha \omega|_{\mathsf{U}\cap\mathsf{U}'} = \mathscr{L}_{\alpha'}\omega|_{\mathsf{U}\cap\mathsf{U}'}$$

*Hence they patch together to a single holomorphic function on the larger neighbourhood $\mathsf{U} \cap \mathsf{U}'$ with opening $\mathsf{copol}(\alpha) \cup \mathsf{copol}(\alpha')$.*

What this means is that, as we vary the ray $\alpha$ through a sector $\mathsf{A}$ of nonsingular directions, we can patch together the different Laplace transforms to get a single function:

**Corollary 1.** *Let $\mathsf{A}$ be a sector of size $|\mathsf{A}| < \pi$ at $p \in \mathsf{X}$. If $\mathsf{A}$ contains no singular directions, then the Laplace transforms $\mathscr{L}_\alpha(\omega)$ for $\alpha \in \mathsf{A}$ assemble into a single analytic function $\mathscr{L}_\mathsf{A}(\omega)$ defined in a sectorial neighbourhood of $p$ with opening $\mathsf{copol}(\mathsf{A})$.*

## 1.2 Laplace transforms in singular directions

We must now understand what happens when the ray $\alpha$ is ***singular***, i.e. it intersects some nonzero points $v_1, v_2, \ldots, \in \Gamma$. We would like to take a Laplace transform along $\alpha$, but we are impeded by the singularities. So we must deform our contour slightly in order to ensure that it misses the singular points. This is always possible because $\Gamma$ is a discrete set.

Indeed, the ray $\alpha$ now determines a pair $\alpha_\pm$ of paths in the Riemann surface $\mathsf{T}_\Gamma$, well-defined up to homotopy, in the following way. At each point, we modify $\alpha$ by introducing a small semi-circle around each point $v_i$. This semicircle could pass on either side of $v_i$. We declare that path $\alpha_+$ is obtained by choosing each of these semicircles to pass to the left of each point, while $\alpha_-$ is obtained by passing to the right, as shown in Figure 4. (Notice that the terms left and right make sense here because the ray $\alpha$ has a definite direction.)

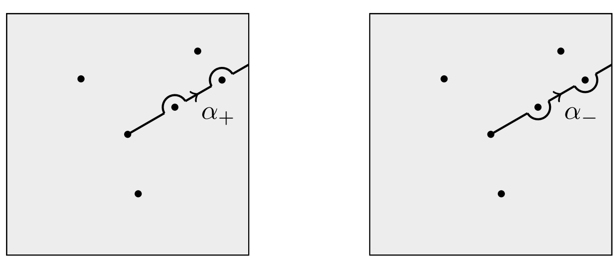


Figure 4: The two ways of lifting a ray in $\mathsf{T}$ to a path in $\mathsf{T}_\Gamma$.

We can now define the ***left and right Laplace transforms*** $\mathscr{L}_{\alpha_+}$ and $\mathscr{L}_{\alpha_-}$ by the formula

$$\mathscr{L}_{\alpha_\pm}(\omega) = \int_{\alpha_\pm} e^{-t/x}\omega$$

The left and right Laplace transforms are sometimes called ***lateral Laplace transforms***. Once again, they define functions in sectorial neighbourhoods with opening $\mathsf{copol}(\alpha)$.

## 1.3 Passing through a singular ray: examples

We would like to understand the difference

$$(\mathscr{L}_{\alpha_-} - \mathscr{L}_{\alpha_+})\omega = \int_{\alpha_- - \alpha_+} e^{-t/x}\omega.$$

of the right and left Laplace transforms along a singular ray. Let us consider some illustrative examples.

*Example* 1. Choose a point $v \in \mathsf{T}$ with coordinate $t(v) = a \in \mathbb{C}$, and consider the form $\frac{dt}{t-a}$, which has an endless continuation away from $v$, and is single valued. Let $\alpha$ be the ray in the direction of $v$; it is the only singular ray. The integral along $\alpha_- - \alpha_+$ clearly reduces to the integral around a small positively oriented loop that encircles the point $v \in \mathsf{T}$:

$$\begin{aligned}(\mathscr{L}_{\alpha_-} - \mathscr{L}_{\alpha_+})f &= \int_{\alpha_- - \alpha_+} \frac{e^{-t/x}\,dt}{t-a} \\ &= 2\pi i \mathrm{Res}_{t=a}\left(\frac{e^{-t/x}\,dt}{t-a}\right) \\ &= 2\pi i \cdot e^{-a/x}\end{aligned}$$

Hence, when the direction of summation passes through $\alpha$, the sum jumps by a function that decays exponentially in the copolar region $\mathsf{copol}(\alpha)$. The exact decay rate depends on how far away $v$ is from the origin. □

*Example* 2. Once again, let $v \in \mathsf{T}$ be a point with coordinate $a = t(v)$, and now let $\omega_0 \in \Omega^1_{\exp}(\mathsf{T})$ be an entire form of exponential type. Consider the form

$$\omega = \frac{\log(t-a)}{2\pi i}\omega_0$$

where we fix once and for all a branch of $\log(t-a)$ at the origin $t = 0 \in \mathsf{T}$. (The choice of branch will be inconsequential in what follows.) Evidently, $\omega$ is $\Gamma$-continuable and the only singular point is $v$. So once again the only singular ray is the ray $\alpha$ in that direction.

We now have

$$(\mathscr{L}_{\alpha_-} - \mathscr{L}_{\alpha_+})\omega = (2\pi i)^{-1}\int_{\gamma_r} e^{-t/x}\log(t-a)\omega_0 + \int_{\beta_{r-}-\beta_{r+}} e^{-t/x}\omega$$

where the contours $\gamma_r$ and $\beta_r$ are indicated in Figure 5.

The contour $\gamma_r$ is easily dispensed with. If $\gamma_r$ is a circle of radius $r$, then along $\gamma_r$, the norm of the integrand is bounded by $C \log r$ for some $C > 0$.

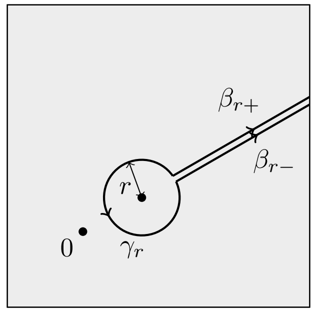


Figure 5: The integration contours $\gamma_r$ and $\beta_{r\pm}$ determined by a small radius $r > 0$.

Hence the integral is at most $2\pi r \cdot C \log r$, which tends to zero as $r$ goes to zero. It remains to see what happens to the integrals along $\beta_{r\pm}$ as $r \to 0$.

But the two rays $\beta_{r\pm}$ lie on different sheets of the Riemann surface. Considering the branching of logarithm as we encircle $\gamma_r$, we see that $\omega|_{\beta_{r-}}$ differs from $\omega|_{\beta_{r+}}$ by the globally defined holomorphic form $e^{-t/x}\omega_0$. We therefore have

$$
\begin{aligned}
(\mathscr{L}_{\alpha_-} - \mathscr{L}_{\alpha_+})\omega &= \lim_{r\to 0} \int_{\beta_{r-}-\beta_{r+}} e^{-t/x}\omega \\
&= \lim_{r\to 0} \int_{\beta_{r\pm}} e^{-t/x}\omega_0 \\
&= \int_{\beta} e^{-t/x}\omega_0,
\end{aligned}
$$

where $\beta$ is the ray from $v$ to $\infty$ in the direction $\alpha$.

This looks integral looks like a Laplace transform of $\omega$, but now the integral starts from $v$ instead of the origin. We can turn it into an honest Laplace transform as follows. Consider the map $\sigma_v : \mathsf{T} \to \mathsf{T}$ defined by the translation $w \mapsto w + v$. Then the contour $\beta$ is the translate of $\alpha$:

$$\beta = (\sigma_v)_*\alpha$$

Meanwhile, we have the pullback

$$\sigma_v^* e^{-t/x} = e^{-(t+t(v))/x} = e^{-(t+a)/x}$$

from which we compute

$$\begin{aligned}(\mathscr{L}_{\alpha_+} - \mathscr{L}_{\alpha_-})\omega_0 &= \int_{(\sigma_v)_*\alpha} e^{-t/x}\omega_0 \\ &= \int_\alpha \sigma_v^* \left(e^{-t/x}\omega_0\right) \\ &= \int_\alpha e^{-(t+a)/x}\sigma_v^*\omega_0 \\ &= e^{-a/x}\mathscr{L}_\alpha(\sigma_v^*\omega_0).\end{aligned}$$

So the jump in the Laplace transform is computed as another Laplace transform, but it is the Laplace transform of a different one-form, which measures the multi-valuedness of $\omega$ at the singular point. Notice that this Laplace transform has been weighted by the factor $e^{-a/x}$, which once again decays in the whole copolar $\mathsf{copol}(\alpha)$. So again the jump is given by an exponentially small term, dictated by the structure of the singularity at $v$. □

## 1.4 Simple singularities

The examples in the previous section show that the effect of certain singularities on the Laplace transform can be easily determined. These are the "simple" singularities, which look locally like

$$\omega = \left(\frac{c}{t - t(v)} + g(t - t(v)) \log(t - t(v)) + h\right) dt \tag{1}$$

where $g$ and $h$ are holomorphic and single-valued, and $c \in \mathbb{C}$. Notice that this local form may depend on the homotopy class of a path from 0 to $v$ along which we perform the analytic continuation. In this course, we will restrict our attention to these simple singularities, since they are slightly easier to manage, but the full theory of resurgence does allow for more general cases.

Let us be more precise about the definition. Let $\Gamma \subset \mathsf{T}$ be a discrete subset, and let $\mathsf{V} \subset \mathsf{T}$ be a small disk around $v \in \Gamma$. We choose $\mathsf{V}$ small enough to ensure that $\mathsf{V} \cap \Gamma = \{v\}$ and $\sigma_{-v}(\mathsf{V}) \cap \Gamma = \{0\}$, where $\sigma_{-v} : \mathsf{T} \to \mathsf{T}$ denotes the translation by $-v$, as shown in Figure 6.

Notice that the preimage $\pi^{-1}(\mathsf{V}) \subset \mathsf{T}_\Gamma$ decomposes into several connected components $\mathsf{V}_\gamma$, each labelled by a homotopy class of paths from 0 to $v$ that avoid $\Gamma$ except at their endpoints. Each component $\mathsf{V}_\gamma$ provides a universal cover of the punctured disk $\mathsf{V} \setminus \{v\}$ via the projection $\pi$. The group of deck transformations of the covering is thus a copy of $\mathbb{Z}$, with generator $\phi \in \mathsf{Aut}(\pi^{-1}(\mathsf{V}))$ given by wrapping once around $v$.

We say that a $\Gamma$-continuable form $\omega \in \Omega^1(\mathsf{T}_\Gamma)$ has ***logarithmic branching at*** $v \in \Gamma$ if for every connected component $\mathsf{V}_\gamma$ there exists a holomorphic form

$$\rho \in \Omega^1(\mathsf{V})$$

such that

$$(\phi^*\omega - \omega)\,|_{\mathsf{V}_\gamma} = \pi^*\rho.$$

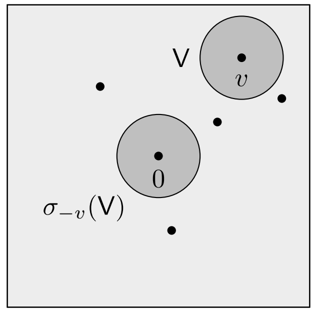


Figure 6: A small disk around a singular point $v$ and its translate.

Clearly, the form $\rho$, if it exists, is uniquely determined by $\omega$ and $\gamma$ via this formula. Its pullback $\omega_\gamma = \sigma_v^*\rho$ along the translation by $v$ is therefore a holomorphic form defined in the disk $\sigma_{-v}(\mathsf{V})$ centred about $0 \in \mathsf{T}$. This form evidently only depends on $\omega$ and $\gamma$, so we denote it by $\omega_\gamma$.

**Definition 1.** The form $\omega_\gamma$ constructed as above is called the ***minor of*** $\omega$ ***along*** $\gamma$.

So the minor $\omega_\lambda$ is a measure of the multi-valuedness of $\omega$ after continuation along the path $\lambda$. If after analytic continuation along $\gamma$, the form $\omega$ has the local structure (1) above, then the minor $\omega_\gamma$ is given by

$$\omega_\gamma = 2\pi i\, g(t)\, dt.$$

**Proposition 2.** *Let $\omega \in \Omega^1(\mathsf{T}_\Gamma)$ be a $\Gamma$-continuable form with logarithmic branching at $v \in \Gamma$. Then for any homotopy class $\gamma$ from $0$ to $v$ avoiding $\Gamma$, the minor $\omega_\gamma$ has an endless continuation away from the translate $\sigma_{-v}(\Gamma)$.*

*Proof.* Consider a connected component $\mathsf{V}_\gamma \subset \pi^{-1}(\mathsf{V})$ as above. Evidently the form $\phi^*\omega$ can be analytically continued along any path in $\mathsf{T} \setminus \Gamma$ because $\omega$ can be continued there. Hence the difference $\phi^*\omega - \omega$ also admits an analytic continuation along any path starting in $\mathsf{V} \setminus \{v\}$ and avoiding $\Gamma$. Translating by $-v$, we obtain the desired property of $\omega_\gamma$. □

If $\omega$ has logarithmic branching after continuation along $\gamma$, then the form

$$\omega - \frac{\log(t - t(v))}{2\pi i}\sigma_{-v}^*\omega_\gamma$$

on $\mathsf{V}_\gamma$, constructed from the minor and a branch of logarithm, will be invariant under deck transformations. It will therefore descend to a form $\mu_\gamma \in \Omega^1(\mathsf{V}\setminus\{0\})$ on the punctured disk.

**Definition 2.** We say that $\omega$ has a ***simple singularity at*** $v$ if $\omega$ has logarithmic branching, and each corresponding form $\mu_\gamma$ has at most a first-order pole at $v$.

Notice that, if we change the linear coordinate $t$, the form $\mu_\gamma$ will be shifted by a multiple of $\omega_\gamma$, but its residue $\mathrm{Res}_v \mu_\lambda \in \mathbb{C}$ will be unchanged; hence we have obtain a second invariant of $\omega$ at $\gamma$: its ***residue***

$$\mathrm{Res}_\gamma \omega \in \mathbb{C}$$

We emphasize that these invariants depend on the full homotopy class $\gamma$, not just the point $v$.

### 1.5 Passing through simple singularities

Suppose that $\omega \in \Omega^1(\mathsf{T}_\Gamma)$ has only simple singularities. Let $\alpha$ be a singular ray. We would like to understand the jump in the Laplace transform of $\omega$ as we pass through $\alpha$. To do this we must integrate along the contour $\alpha_+ - \alpha_-$ as before.

Evidently, this integral has a contribution from every singular point $v_1, v_2, \ldots \in \Gamma$ that the ray $\alpha$ encounters. Calculating exactly as in Section 1.3, we find that each singular point contributes a term of the form

$$\left(2\pi i \mathrm{Res}_{\gamma_v} \omega + \mathscr{L}_{\alpha_+}(\omega_{\gamma_v})\right) e^{-t(v)/x}$$

involving the residue and the minor at the singular point. The residue and minor are extracted by analytically continuing along a path $\gamma$ that goes from 0 to $v$ along $\alpha$, missing every singular point up to $v$ by passing to the right. In this way we obtain the following

**Proposition 3.** *If $\omega$ has only simple singularities along the ray $\alpha$, then*

$$\mathscr{L}_{\alpha_-}\omega - \mathscr{L}_{\alpha_+}\omega = \sum_{v \in \Gamma \cap \alpha} \left(2\pi i \mathrm{Res}_{\gamma_v} \omega + \mathscr{L}_{\alpha_+}(\omega_{\gamma_v})\right) e^{-t(v)/x}$$

Evidently a similar formula holds with the role of the plus and minus signs reversed, in which case, the paths $\gamma_v$ must pass to the left of the other singular points instead of the right.

## 2 Borel sums and resurgent symbols

### 2.1 Borel summation

As we have now alluded to several times, the method of Borel summation gives a way to resum certain divergent series. Indeed, let $f \in \widehat{\mathcal{O}}_{\mathsf{X},p} \cong \mathbb{C}[[x]]$ be a resurgent series, and assume that the Borel transform $\omega = \widehat{\mathscr{B}}(f)$ has exponential type at infinity.

Suppose that $\alpha$ is a nonsingular ray. Then we define the ***Borel sum of $f$ in the direction*** $\alpha$ by the formula

$$s_\alpha f = f(p) + \mathscr{L}_\alpha \widehat{\mathscr{B}}(f) = f(p) + \int_\alpha e^{-t/x}\omega$$

where $f(p)$ denotes the constant term of $f$. On the other hand, if $\alpha$ is a singular direction, then can define left and right Borel sums $s_{\alpha_\pm} f$ along singular rays, but they will, in general, be different. This jump in the Borel sum as we pass through a singular point is the ***Stokes phenomenon***.

Let us focus for the moment on the nonsingular case. As we saw in Corollary 1, if $\mathsf{A} \subset \mathsf{S}_p\mathsf{X}$ is a sector that contains no singular directions, then we can patch together the sums along the different rays in $\mathsf{A}$, and define the Borel sum in the direction $\mathsf{A}$ to be the function

$$s_{\mathsf{A}} f = f(p) + \mathscr{L}_{\mathsf{A}} \widehat{\mathscr{B}}(f).$$

We then have the following

**Theorem 1.** *Let $f \in \widehat{\mathcal{O}}_{\mathsf{X},p}$ be a summable resurgent series whose Borel transform has exponential type, and let $\mathsf{A} \subset \mathsf{S}_p\mathsf{X}$ be a sector of size $|\mathsf{A}| < \pi$ that contains no singular directions of $f$. Then the Borel sum $s_{\mathsf{A}} f$ is a Gevrey function that is asymptotic to $f$ in a sectorial neighbourhood of $p$ with opening* $\mathsf{copol}(\mathsf{A})$.

To prove this result, one must give appropriate bounds on the derivatives of $f$; this is done by differentiating under the integral sign using the formula for the Laplace transform, and then integrating by parts. We shall not go into details since this argument is similar to the discussion in Section 2.4 of Lecture 3. See, for example, [1, Section 7] for a proper treatment.

One key observation is that, by construction, the copolar $\mathsf{copol}(\mathsf{A})$ has an opening angle $|\mathsf{copol}(\mathsf{A})| > \pi$. Hence there can be at most one Gevrey function that is asymptotic to $f$ in this sector. What this means it that the Borel sum $s_{\mathsf{A}} f$ is, in some sense, the unique sum of the series in this region. (Although there will of course be other sums that are not Gevrey.) In particular, although the Borel and Laplace transforms depend on the choice of a coordinate $x$, the Borel sum $s_{\mathsf{A}} f$ is ultimately independent of this choice.

## 2.2 Algebras of resurgent symbols

In order to deal with the Stokes phenomenon, we must understand how the sum of the series jumps when we cross through a singular point. For this it will be useful to focus our attention on the case of simple singularities:

**Definition 3.** A ***simple resurgent function*** is a Gevrey formal power series $f \in \mathbb{C}[[x]] = \mathcal{O}_{\mathsf{X},p}$ whose Borel transform admits an endless analytic continuation with only simple singularities.

We must then account for the fact that expressions of the form $g\, e^{-a/x}$ will start appearing once we pass through singular directions. The series $g$ that appear in this way are always given by Laplace transforms of minors $\omega_\gamma$. Since the minors are endlessly continuable with simple singularities, these new series $g$ are also simple resurgent functions.

**Definition 4.** An ***elementary resurgent symbol*** is an expression of the form $ge^{-a/x}$ where $g \in \mathbb{C}[[x]] = \widehat{\mathcal{O}}_{\mathsf{X},p}$ is a resurgent series. The ***support*** of an elementary resurgent symbol is the vector $v \in \mathsf{T}$ such that $t(v) = a$.

When we pass through a singular ray $\alpha$, the Borel sum jumps by a series of elementary resurgent symbols that all decay in the sector defined by $\alpha$. We would like to allow $\alpha$ to vary in a small sector and collect all such expressions into a single algebra, so we make the following definition:

**Definition 5.** Let $\mathsf{A} \subset \mathsf{S}_p\mathsf{X}$ be an arc of angle $|\mathsf{A}| < \pi$. A ***resurgent symbol along*** $\mathsf{A}$ with support in the discrete set $\Gamma \subset \mathsf{T}$ is a series

$$f = \sum_{v \in \Gamma} f_v \, e^{-t(v)/x}$$

such that each exponential $e^{-t(v)/x}$ decays as $x \to 0$ in $\mathsf{A}$, i.e. every directions in $\Gamma$ is contained in the polar $\mathsf{pol}(\mathsf{A})$. We denote by $\mathcal{R}(\mathsf{A})$ the space of all resurgent symbols along $\mathsf{A}$.

*Exercise* 1. Determine what happens to an elementary resurgent symbol when we change the coordinate from $x$ to $u = u(x)$. Conclude that elementary resurgent symbols with support $v \in \mathsf{T}_p\mathsf{X}$ should really be viewed as sections of a line bundle over $\mathsf{X}$. This bundle is canonically trivialized away from $p$. Its holomorphic sections can then identified with the holomorphic functions $f$ on $\mathsf{X} \setminus \{p\}$ such that $e^{a/x} f$ is holomorphic at $p$, a condition that is independent of the coordinate $x$. This construction is the analogue for exponential singularities of the well-known line bundles $\mathcal{O}_{\mathsf{X}}(k \cdot p)$ formed by meromorphic functions with poles of order at most $k$ at $p$. $\square$

# References

[1] D. Sauzin, *Introduction to 1-summability and resurgence*, 1405.0356.