Resurgence in Geometry and Physics

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Lecture 4

Abstract

We introduce the convolution product on one-forms that corresponds, via the Borel and Laplace transforms, to the usual product on functions. We then introduce Écalle's notion of endless analytic continuability, and show that two endlessly continuable forms have endlessly continuable convolutions.

Contents

1 Motivation

In the previous lecture, we introduced the Borel transform $\mathscr{B}$ and the Laplace transform $\mathscr{L} = \mathscr{B}^{-1}$, which related holomorphic functions defined near a point p on a Riemann surface X, and holomorphic one-forms on a one-dimensional vector space T (the tangent space to X at p). We also introduced "formal" version of these operations, that act on formal power series.

These operations behaved well with respect to the algebraic structure on functions. For example, they are linear, and there is a natural compatibility with differentiation, given by the formula

$$\mathscr{B}(x^2\partial_x f) = t\mathscr{B}(t)$$

Our task in this lecture is describe the compatibility with multiplication with functions, which is more subtle. It will naturally lead us to certain infinite-sheeted Riemann surfaces and Écalle's notion of endless analytic continuation.

2 The convolution product

2.1 Definition of convolution

Recall that if x is a coordinate on the Riemann surface X at p, and t is a global linear coordinate on our vector space T, the formal Borel transform is given by

$$\widehat{\mathscr{B}}\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!} dt.$$

It takes a power series in X with zero constant term and produce a one-form on T with formal power series coefficients. Now, formal series have the usual multiplication law corresponding to multiplication of functions on X. But one-forms on T do not have an obvious product; normally forms cannot be multiplied. On the other hand, since $\widehat{\mathscr{B}}$ is an isomorphism, we can transport the product of functions on X to a product on forms on T. The result is the ***convolution product***

$$\omega * \mu = \widehat{\mathscr{B}}(\widehat{\mathscr{L}}(\omega)\widehat{\mathscr{L}}(\mu))$$

where $\widehat{\mathscr{L}} = \widehat{\mathscr{B}}^{-1}$ is the formal Laplace transform. By linearity, this product is determined completely by what it does to elements of the form $t^n\, dt$ with $n \geq 0$, for which it is easily computed:

$$\begin{aligned}(t^j\, dt) * (t^k\, dt) &= \widehat{\mathscr{B}}(\widehat{\mathscr{L}}(t^j) \cdot \widehat{\mathscr{L}}(t^k)) \\ &= \widehat{\mathscr{B}}(j!x^{j+1} \cdot k!x^{k+1}) \\ &= j!k!\widehat{\mathscr{B}}(x^{j+k+2}) \\ &= \frac{j!k!}{(j+k+1)!} t^{j+k+1}\, dt.\end{aligned}$$

In fact, this product may be constructed without reference to the transforms $\widehat{\mathscr{B}}$ and $\widehat{\mathscr{L}}$, using only the fact that T is a vector space. To this end, suppose that ω and μ are holomorphic forms defined in some disk $\mathsf{V} \subset \mathsf{T}$ containing zero. Thus we can write $\omega = f\, dt$ and $\mu = g\, dt$ for some holomorphic functions f and g defined on V. Then the convolution product is given by

$$\omega * \mu = h\, dt$$

where

$$h(t) = \int_0^t f(\tau)g(t-\tau)d\tau = t\int_0^1 f(t\tau)g((1-\tau)t)\,d\tau \tag{1}$$

and the integral is taken along a straight line path from 0 to t in V.

Exercise 1. Prove that this integral formula reproduces the correct formula for the convolution of $t^j\,dt$ and $t^k\,dt$. □

Remark 1. The product may also be phrased in a coordinate-free manner using an appropriate pushforward on differential forms; see Appendix A. □

2.2 An example: powers of Euler's series

Let us return to our example of the Euler series

$$f = \sum_{k=0}^{\infty} (-1)^k k! x^{k+1}$$

from Lecture 1. Its Borel transform is given by

$$\omega = \widehat{\mathscr{B}}(f) = \sum_{k=0}^{\infty}(-1)^k t^k\,dt = \frac{dt}{1+t}$$

Here, we have summed the series for $|t| < 1$, and then extended the resulting form to all of T by analytic continuation. In so doing, we found that ω has a pole at $t = -1$, but it is otherwise single-valued and holomorphic. In other words, the singular set of ω is given by

$$\mathsf{sing}(\omega) = \{t = -1\} \subset \mathsf{T}.$$

Suppose that we wanted to know the Borel transform of f^2. In light of our considerations above, this means that we should look at the convolution product

$$\omega^{*2} = \omega * \omega = \frac{dt}{1+t} * \frac{dt}{1+t}$$

Using the integral formula for the convolution, and partial fraction decomposition, we can easily compute the convolution:

$$\begin{aligned}\int_0^t \frac{1}{1+\tau}\frac{1}{1+(t-\tau)}d\tau &= \frac{1}{t+2}\int_0^t\left(\frac{d\tau}{t-\tau+1} + \frac{d\tau}{\tau+1}\right)\\ &= \frac{1}{t+2}\left(-\log(t-\tau+1) + \log(\tau+1)\right)\Big|_{\tau=0}^{\tau=t}\\ &= \frac{2\log(t+1)}{t+2}\end{aligned}$$

We therefore have the simple formula

$$\omega^{*2} = \widehat{\mathscr{B}}(f^2) = \frac{2\log(t+1)}{t+2}dt.$$

We see immediately that the convolution has produced a new singularity: there is now a pole at $t = -2$. But there is a further complication: the form is now multivalued due the factor $\log(t+1)$, which has a branch point at $t = -1$, where we originally had the pole. So now the singular set

$$\mathsf{sing}(\omega^{*2}) \subset \mathsf{T}$$

consists of the two points $t = -1$ and $t = -2$.

Let us denote by T_2 the universal cover of $\mathsf{T} \setminus \mathsf{sing}(\omega^{*2})$. Recall that in general, the construction of the universal cover depends on the choice of base point, but here we have a canonical choice given by the origin $0 \in \mathsf{T}$. Likewise ω^{*2} has a preferred branch at the origin, determined by the standard branch of logarithm. In this way, we see that ω^{*2} is a canonically defined, single-valued, holomorphic one-form on the Riemann surface T_2. This Riemann-surface has infinitely many sheets over our original space T.

Continuing in this way to compute the next power, we find

$$\omega^{*3} = \Big(\mathrm{Li}_2\left(\tfrac{1}{t+2}\right) + \mathrm{Li}_2(-t-1) - \mathrm{Li}_2\left(\tfrac{t+1}{t+2}\right) \\ +2\log(t+2)\log(t+1) + \frac{\pi^2}{12}\Big)\,\frac{2dt}{t+3}$$

where

$$\mathrm{Li}_2(u) = -\int_0^u \frac{\log(1-w)}{w}dw = \sum_{k=1}^{\infty}\frac{u^k}{k^2}$$

is the dilogarithm function. Therefore the singular set of the one-form ω^{*3} is given by the three points where $t = -1, -2$ and -3. Thus ω^{*3} is defined on the universal cover T_3 of $\mathsf{T} \setminus \mathsf{sing}(\omega^{*3})$.

A clear pattern is emerging. The singularity at $t = -1$ in ω is propagating: every time we apply the convolution, we end up with a new singularity that is shifted by -1 from the previous ones, so that the form ω^{*n} has singularities at $t = -1, -2, \ldots, -n$. We obtain a form that is defined on the infinite-sheeted Riemann surface T_n, which is the universal cover of $\mathsf{T} \setminus \mathsf{sing}(\omega^{*n})$. In order to work with all possible powers of ω together, we should take the universal cover T_∞ of $\mathsf{T} \setminus \{-1, -2, -3, \ldots\}$.

The fact that the singularities of $\omega^{*(i+j)}$ are obtained by adding together the singularities of ω^{*i} and ω^{*j} is no coincidence, as we will soon explain.

3 Endless analytic continuation and resurgent functions

3.1 Endless analytic continuation

We now come to one of the key ideas of resurgence theory: Écalle's notion of endless analytic continuation. We will use here only a simplified version, following [1, 2].

We want to begin with the germ of a one-form defined in a neighbourhood of the origin in T, and extend it analytically to all of T. We must therefore consider paths γ that start at 0 and try to analytically continue the form along γ. In so doing, we may find that there are certain points in T that we must avoid. But if we are lucky, these points will form a discrete subset $\Gamma \subset \mathsf{T}$ where the singularities are poles or branch points. In this case we say that the form has an ***endless analytic continuation***.

More precisely, let $\Gamma \subset \mathsf{T}$ be a (possibly infinite) discrete subset of T. In general, this subset may or may not contain $0 \in \mathsf{T}$, but we will always assume that $0 \in \Gamma$ as we will lose no generality by doing so. A ***path from the origin that avoids*** Γ is a smooth path $\gamma : [0,1] \to \mathsf{T}$ such that

1. $\gamma(0) = 0$
2. $\gamma(t) \in \mathsf{T} \setminus \Gamma$ for $t > 0$

Two such paths are ***homotopic*** when they can be joined by a homotopy that fixes the endpoints and avoids Γ in the same way; see Figure 1.

Definition 1. For a discrete set $\Gamma \subset \mathsf{T}$, let

$$\mathsf{T}_\Gamma^\circ = \frac{\{\text{paths from the origin avoiding } \Gamma\}}{\text{homotopies}},$$

and let

$$\mathsf{T}_\Gamma = \mathsf{T}_\Gamma^\circ \cup \{\hat{0}\},$$

where $\hat{0}$ denotes the constant path at the origin in T. We denote by

$$\begin{array}{cccc} \pi_\Gamma & : & \mathsf{T}_\Gamma & \to & \mathsf{T} \\ & & [\gamma] & \mapsto & \gamma(1) \end{array}$$

the map that extracts the endpoint of a path.

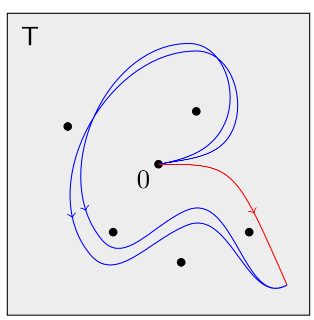

Figure 1: Some paths avoiding the discrete set Γ, denoted by the black dots. The two blue paths are homotopic, but they are not homotopic to the red one.

Notice, in particular, that

$$\pi_\Gamma(\mathsf{T}_\Gamma^\circ) = \mathsf{T} \setminus \Gamma$$

and

$$\pi_\Gamma(\hat{0}) = 0.$$

Moreover, if $\mathsf{V} \subset \mathsf{T}$ is a simply-connected open neighbourhood of zero such that $\mathsf{V} \cap \Gamma = \{0\}$, then by considering the paths from the origin that are completely contained in V, we obtain a canonical embedding $\mathsf{V} \subset \mathsf{T}_\Gamma$ that sends $0 \in \mathsf{V}$ to $\hat{0} \in \mathsf{T}_\Gamma$. We will identify V with its image in both T and T_Γ, the choice being clear from context.

The following statement is a straightforward consequence of these observations, and the usual arguments regarding the construction of the universal cover of a manifold:

Proposition 1. *For a discrete subset $\Gamma \subset \mathsf{T}$, there is a unique Riemann surface structure on T_Γ such that the map $\pi_\Gamma : \mathsf{T}_\Gamma \to \mathsf{T}$ is locally a biholomorphism. Moreover, T_Γ is simply connected.*

Definition 2. Let $\Gamma \subset \mathsf{T}$ be a discrete subset containing the origin, and let $\omega \in \Omega^1(\mathsf{V})$ be a one-form defined in a neighbourhood $\mathsf{V} \subset \mathsf{T}$ of the origin. We say that ω is ***Γ-continuable*** if it extends to a holomorphic form on the entire Riemann surface T_Γ by analytic continuation.

Definition 3. A one-form ω defined in a neighbourhood of the origin in T has an ***endless analytic continuation*** if there exists a discrete subset $\Gamma \subset \mathsf{T}$ such that ω is Γ-continuable. A formal power series $f \in \mathbb{C}[[x]]$ is ***resurgent*** if it is Gevrey, and its Borel transform has an endless analytic continuation.

Example 1. The form $\omega = \frac{dt}{1+t}$ obtained as the Borel transform of the Euler series $f = \sum_{k=0}^\infty (-1)^k k! x^{k+1}$ is Γ-continuable, where $\Gamma = \{t = 0, -1\}$. Therefore the Euler series f is resurgent. In fact, ω is already defined on $\mathsf{T} \setminus \{t = -1\}$ and is thus extended to T_Γ simply by pulling back along π_Γ. □

Example 2. The series

$$\omega = \sum_{k=0}^\infty \frac{t^k\, dt}{k+1}$$

defines a holomorphic form in the disk $|t| < 1$. We claim that it has an endless analytic continuation. Indeed, it is Γ-continuable, where $\Gamma = \{t = 0, 1\} \subset \mathsf{T}$. Indeed we have for $|t| < 1$ that

$$\Omega = -\mathrm{Log}(1-t)\frac{dt}{t},$$

where Log denotes the principle branch of logarithm. The function $\mathrm{Log}(1-t)$ is holomorphic and vanishes at $t = 0$, so that the ratio $t^{-1}\mathrm{Log}(1-t)$ is holomorphic. Written in this way, it is clear that the form can analytically continued; it has an infinite branch point of log type at $t = 1$. If we analytically continue this form along a path that wraps n times around the branch point and returns to the origin, we find the form

$$\omega_n = -(\mathrm{Log}(1-t) + 2\pi i n)\frac{dt}{t}$$

Thus on the other sheets of the Riemann surface T_Γ, this form has a pole at the origin. This example shows why it is helpful to include the origin in the discrete set Γ. $\square$

Example 3 ([2, p. 26]). Recall Stirling's approximation

$$\log(\Gamma(z)) - \tfrac{1}{2}\log(2\pi) - (z - \tfrac{1}{2})\log z + z \sim \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)z^{2k-1}} =: f$$

for the classical Γ function (not to be confused with the discrete sets $\Gamma \subset \mathsf{T}$ that we have been considering). Here, the coefficients B_{2k} are Bernoulli numbers, and the formula defines an asymptotic expansion as $z \to +\infty$, i.e. the right hand side defines an asymptotic expansion $f \in \mathbb{C}[[x]]$ in the coordinate $x = z^{-1}$ at infinity.

The Borel transform may be explicitly computed using the generating series

$$\tfrac{1}{2}\coth(t/2) = \frac{1}{t} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} t^{2k-1}.$$

for the Bernoulli numbers, where coth is the hyperbolic tangent. Indeed, a straightforward calculation using the identity $\widehat{\mathscr{B}}(x^2\partial_x f) = t\widehat{\mathscr{B}}(f)$ yields the formula

$$\omega = \widehat{\mathscr{B}}(f) = \frac{t\coth(t/2) - 2}{2t^2}\,dt.$$

This form has poles for $t \in 2\pi i\mathbb{Z}$ but is otherwise single-valued and holomorphic. Hence ω has endless an endless continuation, so the Stirling series is resurgent.

The exponentiation of the series reads

$$\sqrt{\frac{z}{2\pi}}\left(\frac{e}{z}\right)^z \Gamma(z) \sim \exp(f) = \sum_{k=0}^{\infty} \frac{f^k}{k!}$$

Once again, the series on the right hand side is resurgent, and since the Borel transform kills the constant term, we expect the Borel transform to be given by

$$\widehat{\mathscr{B}}(e^f - 1) = \omega + \tfrac{1}{2}\omega * \omega + \tfrac{1}{3!}\omega * \omega * \omega + \cdots$$

Examining this formula and comparing with our previous calculations, we expect the singularities of ω to propagate and produce infinitely many branch points. Indeed this is exactly what happens, with branching where $t \in \pi i\mathbb{Z}$. Justifying these assertions requires some work; we refer the reader to [2] for a complete treatment. $\square$

Example 4. The series

$$g(t) = \sum_{k=0}^{\infty} t^{2^k} = 1 + t + t^2 + t^4 + t^8 + \cdots$$

clearly converges to define a holomorphic function on the disk $|t| < 1$. Notice that

$$g(1) = 1 + 1 + 1 + 1 + \cdots ,$$

so this function has a singularity at $t = 1$. But we also have the functional equation

$$g(t^2) = g(t) - t,$$

and hence g must also have a singularity $t = -1$. Continuing in this way with $g(t^{2k}) = g(t^k) - t^k$ for $k \geq 2$, we see that g has a singularity at every point where $t^{2k} = 1$. Such points are dense in the unit circle $|t| = 1$, so the function g cannot be extended beyond this disk. (It is an example of a ***lacunary function***, which means that it has the unit circle as its natural boundary.) Thus, the one-form $g\,dt$ does not admit an endless analytic continuation. □

These examples point to two key features of endless continuation and resurgent series:

1. Not every Gevrey series is resurgent. Indeed, any series whose Borel transform is a lacunary function will not be resurgent. Thus resurgence is not a property that can simply be read off from the growth of the coefficients; it must be established by some other means. Typically, it will be a consequence of some additional property that the series satisfies; for example, the series may solve an analytic differential equation, which puts strong constraints on the relations between the coefficients.

2. Even if we start with series whose Borel transforms are single-valued, or have only a couple of singular points, we quickly find many more singularities and branch points when we start performing natural operations on these series. So working with these infinite-sheeted Riemann surfaces is absolutely essential if we want to reflect the full algebraic structure of the series in question.

3.2 Continuation of convolution products

The following fundamental result shows that resurgent series form an algebra. We shall only sketch the argument here, and refer the reader to [2, Sections 19–21] for more details.

Theorem 1. *Suppose that $\Gamma \subset \mathsf{T}$ is a discrete subset containing zero. Then the space $\Omega^1(\mathsf{T}_\Gamma)$ of Γ-continuable forms is closed under convolution if and only if Γ is closed under addition.*

To see that closure under addition is necessary, consider the convolution of two forms with simple poles at a and b, generalizing our calculations with the Euler series. We find

$$\begin{aligned}\frac{dt}{t-a} * \frac{dt}{t-b} &= \left(\int_0^t \frac{1}{\tau - a}\frac{1}{t-\tau-b}d\tau\right) dt \\ &= (\log(1-t/a) + \log(1-t/b))\frac{dt}{t-(a+b)}\end{aligned}$$

which has a pole at $t = a+b$. So if $a, b \in \Gamma$, we need $a+b \in \Gamma$ as well.

So the interesting part is to show that if Γ is closed under addition, then any pair $\omega, \mu \in \Omega^1(\mathsf{T}_\Gamma)$ of Γ-continuable forms must have a Γ-continuable convolution. That is, if we define their convolution in a small disk $\mathsf{V} \subset \mathsf{T}$ centred at 0 using (1), then we may analytically continue the convolution along any path in $\mathsf{T} \setminus \Gamma$ that starts in V.

The first step is to observe that, while the convolution in V is defined by integrating along rays, it can also be defined by integration along more general paths. Indeed, if $C : [0,1] \to \mathsf{V}$ is a path from 0 to $C(1) = t$ such that $t - C$ also lies in V, then the coefficient of the convolution product may be written as

$$h(t) = \int_0^1 f(C(\tau)) \cdot g(t - C(\tau)) \cdot \gamma'(\tau)\, d\tau, \tag{2}$$

which evidently recovers (1) when C is a ray. One way to ensure that the path $t - C$ will lie in V is to require that C lies in V and satisfies the equation

$$C(1-\tau) = t - C(\tau)$$

for all $\tau \in [0,1]$. For then the two paths are obtained C and $t-C$ are obtained from one another simply by reversing the direction of time. Such a path C is called a ***symmetric path*** from 0 to t, because it has 180° rotational symmetry about its midpoint $C(\frac{1}{2}) = \frac{C(1)}{2}$, as shown in Figure 2.

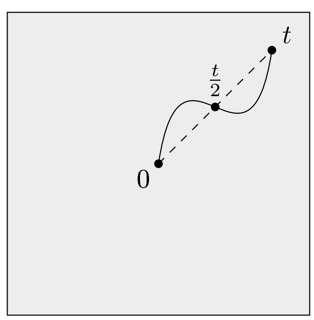

Figure 2: A symmetric path from the origin to $t \in \mathsf{T}$

Now suppose that γ is a path in $\mathsf{T} \setminus \Gamma$ that starts in V. To extend the convolution product along γ it is enough to show that for every $s \in [0,1]$ we can find a symmetric path C_s from 0 to $\gamma(s)$ with the following two properties:

1. C_s avoids Γ, so that it lies entirely in the domain of ω and μ, and
2. C_s depends smoothly on s.

The situation is illustrated in Figure 3. Once such a family is obtained, the convolution may be defined at the point $t = t_s = \gamma(s) \in \mathsf{T} \setminus \Gamma$ using the symmetric path $C = C_s$ to compute the integral (2). It is then easy to see that the result is really an analytic continuation of the convolution in V.

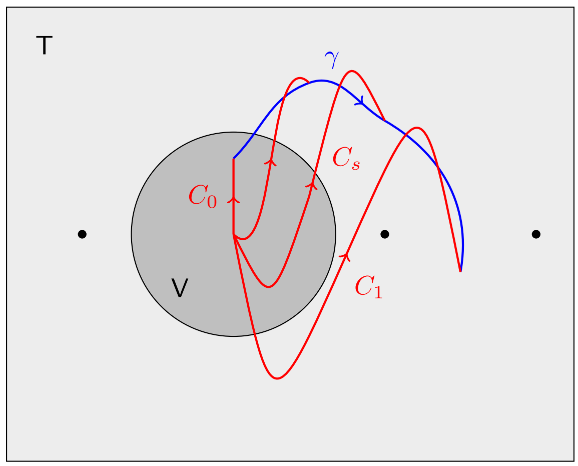

Figure 3: A family of symmetric paths, shown in red, whose endpoints follow the blue curve γ that starts in the disk V. The family starts with the ray C_0 and avoids the points of Γ, denoted by the black dots.

In light of these considerations, the theorem essentially reduces to the following geometric fact:

Lemma 1. *Suppose that $\Gamma \subset \mathsf{T}$ is a discrete set that is closed under addition, and that γ is a path in $\mathsf{T} \setminus \Gamma$. Let C_0 be the ray from 0 to $\gamma(0)$, and assume that C_0 avoids Γ. Then there is a smooth family of symmetric paths $C_s : [0,1] \to \mathsf{T}$ starting from C_0 and parametrized by $s \in [0,1]$, such that C_s avoids Γ for all s, and C_s has $\gamma(s)$ as its endpoint.*

Proof. We construct the paths C_s by flowing the original ray C_0 along an s-dependent vector field Z_s. To define this vector field, we choose a smooth function $G : \mathsf{T} \to [0,1]$ that has Γ as its zero locus, i.e. $G^{-1}(0) = \Gamma$. We then define the vector field Z by the formula

$$Z_s(v) = \frac{G(v)}{G(v) + G(\gamma(s) - v)} \cdot \gamma'(s)$$

We claim that the denominator $G(v) + G(\gamma(s) - v)$ is nonvanishing, so that this vector field is globally well defined. Indeed, if it were to vanish, then since G

takes on only nonnegative values, we must have $G(v) = G(\gamma(s) - v) = 0$. But then v and $\gamma(s) - v$ must both lie in Γ. Since Γ is closed under addition, this would imply that $\gamma(s) \in \Gamma$ as well, contradicting the assumption that γ lies in $\mathsf{T} \setminus \Gamma$.

We observe that with respect to any norm $|\cdot|$ on T, we have $|Z_s(v)| \leq |\gamma'(s)|$ for all $v \in \mathsf{T}$. Hence this s-dependent vector field is complete, i.e. its flow exists for all $s \in [0, 1]$. Applying the flow to the ray C_0, we obtain a family of paths C_s that avoid Γ. The fact that these paths remain symmetric now follows easily from the identity

$$Z_s(v) + Z_s(\gamma(s) - v) = \gamma'(s)$$

and the fact that C_0 is symmetric. $\square$

A Geometric description of convolution

In this appendix, we give a more geometric construction of the convolution product. The construction uses the diagram

$$\begin{tikzcd} & \mathsf{T} \times \mathsf{T} \arrow[r, "a"] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \mathsf{T} \\ \mathsf{T} & & \mathsf{T} \end{tikzcd}$$

where p_1, p_2 are the projections and $a : \mathsf{T} \times \mathsf{T} \to \mathsf{T}$ denotes addition in the vector space T.

Given one-forms $\omega, \mu \in \Omega^1(\mathsf{T})$, consider the two-form

$$p_1^*\omega \wedge p_2^*\mu \in \Omega^2(\mathsf{T} \times \mathsf{T}).$$

We claim that the convolution is given by a "pushforward"

$$\omega * \mu = a_*(p_1^*\omega \wedge p_2^*\omega) \in \Omega^1(\mathsf{T}),$$

defined by integrating the two-form over a path in each fibre of a, to get a one-form on T.

Indeed, suppose that $v \in \mathsf{T}$, and that $w \in \mathsf{T}_v\mathsf{T} \cong \mathsf{T}$ is a tangent vector at v. We wish to define the pairing

$$\langle (\omega * \mu)|_v, w \rangle \in \mathbb{C}$$

To this end, let $\gamma : [0, 1] \to \mathsf{T}$ be a path from the origin to v in T. Then γ defines a path $\tilde{\gamma}$ in the fibre

$$a^{-1}(v) = \{(u, v - u) \in \mathsf{T} \times \mathsf{T} \,|\, u \in \mathsf{T}\}$$

by the formula $\widetilde{\gamma} = (\gamma, v - \gamma)$. Indeed, every path in the fibre has this form.

Likewise, we can consider the constant vector field

$$\widetilde{w} = \tfrac{1}{2}(w, w) \in \mathsf{T} \times \mathsf{T}$$

on $\mathsf{T} \times \mathsf{T}$, which projects to w along a. Then the ***convolution of*** ω ***and*** μ ***along*** γ is given by

$$\langle (\omega *_\gamma \mu)|_v, w\rangle = \int_{\widetilde{\gamma}} \iota_{\widetilde{w}}(p_1^*\omega \wedge p_2^*\mu) \tag{3}$$

where $\iota_{\widetilde{w}}$ denotes the contraction of the vector field $\widetilde{w}$ into forms.

Exercise 2. Show that the formula (3) for the convolution along a ray γ directly recovers the coordinate expression (1). □

Now suppose that ω and μ, rather than being globally defined, are defined only in some open set $\mathsf{V} \subset \mathsf{T}$ containing $0, v \in \mathsf{T}$. Then (3) makes sense as long as $\gamma : [0,1] \to \mathsf{V}$ is a path from 0 to v with the property that $v - \gamma$ is also a path in V. Let us say that such a path is ***adapted to*** V.

Exercise 3. Suppose that γ_s for $s \in [0,1]$ is a smooth family of paths from 0 to v adapted to V. Show that the convolution

$$\omega *_{\gamma_s} \mu$$

is independent of s. Hence, the convolution along a path adapted to V depends only on the homotopy class of the path, where the homotopies are also adapted to V. □

References

[1] D. Dorigoni, *An Introduction to Resurgence, Trans-Series and Alien Calculus*, 1411.3585.

[2] D. Sauzin, *Introduction to 1-summability and resurgence*, 1405.0356.