Resurgence in Geometry and Physics

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Lecture 2

Abstract

We introduce the notion of a real oriented blowup as convenient way to study functions defined in angular sectors in Riemann surfaces. We examine holomorphic functions that have exponential growth in sectors and prove the Fragmen–Lindelöf principle. We then recall Poincaré's notion of an asymptotic expansion, and show that every formal power series is the asymptotic expansion of a holomorphic function defined in a sector.

Contents

1 The importance of directions

Recall that our aim is to study functions defined by expansions such as

$$f = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{jk}x^k e^{c_j/x}.$$

They have two main pieces: essential singularities due to the exponentials, and power series expansions that are typically divergent. Our goal in this lecture is to look at these two pieces individually.

Before we begin in earnest, let us recall that in the previous lecture, we solved the differential equation

$$x^2 f' = x - f$$

using the series

$$f = \sum_{k=0}^{\infty} (-1)^k k! x^{k+1},$$

We were able to "resum" the series for to get a solution of the ODE in the complex domain $x \in \mathbb{C}$, but there was a caveat: there was a certain special direction in which we could not resum the series, namely the negative real axis $x < 0$. We found that along this axis, the function had a branch cut, and that the two branches differed by a multiple of the function

$$g = e^{1/x},$$

which has an essential singularity at the origin. This was an example of a general rule:

> **The sum of a divergent series will depend on the "direction" in which take the sum.**

But functions with essential singularities also often have characteristic directions. Indeed, let us consider the function g above. Let us use polar coordinates

$$x = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

so that

$$g = \exp\left(\frac{\cos\theta - i\sin\theta}{r}\right)$$

In particular,

$$|g| = \exp(r^{-1}\cos\theta).$$

Imagine that $r \to 0$ along a ray of fixed angle θ. Then there are three quite distinct possibilities:

1. $\cos\theta > 0$, in which case $g \to \infty$ very rapidly as $r \to 0$
2. $\cos\theta < 0$, in which case $g \to 0$ very rapidly as $r \to 0$
3. $\cos\theta = 0$, in which case $g = \exp(\pm i/r)$ has unit modulus, but oscillates very wildly as $r \to 0$.

So the plane is naturally divided into two open regions—the left and right half-planes—on which the function grows or decays very rapidly. These regions are bounded by rays on which the function oscillates wildly. More generally, the behaviour of the function $\exp(1/x^k)$ with $k \in \mathbb{Z}_{>0}$ divides the plane into sectors of total angle π/k, and the function alternates between growing rapidly and decaying rapidly in each of these regions, as shown in Figure 1.

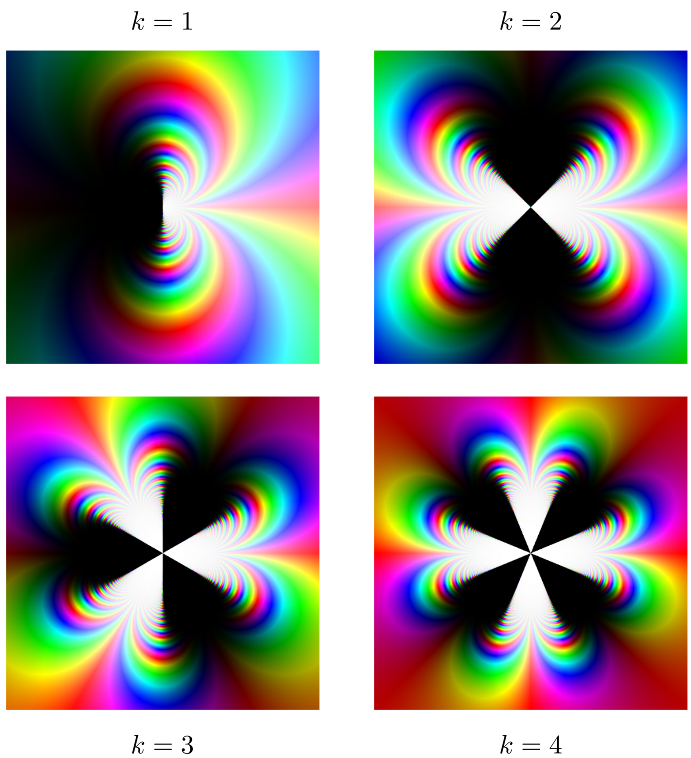

Figure 1: Contour plots of the functions $f = e^{1/x^k}$ for $1 \leq k \leq 4$. The magnitude is indicated by the brightness, while the phase is indicated by the colour.

We would like to think of these sectors as being like open neighbourhoods of the point p in which we have some understanding of the function. But as it stands, such sectors are not really open neighbourhoods in the traditional sense, as none of them contains a ball centred at p. What's happening is that the sector defined by a condition $\theta \in (a, b)$ on the angular coordinate, but this condition does not define an open neighbourhood of 0, because the polar coordinate representation

$$x = re^{i\theta}$$

is ambiguous when $r = 0$. To resolve this confusion, we now introduce the notion of a real-oriented blowup.

2 Real oriented blowups

2.1 Tangent rays

Let X be a Riemann surface and let p be a point in X. We denote by $\mathsf{T}_p\mathsf{X}$ the tangent space of X at p. A ***tangent ray at*** p is a subset $\alpha \subset \mathsf{T}_p\mathsf{X}$ of the form

$$\alpha = \{rv \in \mathsf{T}_p\mathsf{X} \,|\, r > 0\}$$

where $v \in \mathsf{T}_p\mathsf{X}$ is some fixed nonzero tangent vector. We denote by $\mathsf{S}_p\mathsf{X}$ the set of tangent rays at p. If we choose a coordinate x at p, then every ray will point in the direction of a unique unit modulus number $x = e^{i\theta}$, and so $\mathsf{S}_p\mathsf{X}$ is (non-canonically) isomorphic to the standard circle

$$\mathsf{S}_p\mathsf{X} \cong \mathsf{S}^1 = \left\{ e^{i\theta} \,\middle|\, \theta \in [0, 2\pi) \right\}$$

Notice that if we change the coordinate to

$$u = ae^{i\mu}x + bx^2 + \cdots$$

with $a > 0$, then the identification of $\mathsf{S}_p\mathsf{X}$ with S^1 undergoes a rotation by the angle μ. Hence there is no privileged direction "$\theta = 0$" at p, but the term "clockwise direction" still makes sense, and we can still talk about the angle formed by two rays $\alpha_1, \alpha_2 \in \mathsf{S}_p\mathsf{X}$: starting from α_1, we move around $\mathsf{S}_p\mathsf{X}$ counter-clockwise until we reach α_2. In so doing we always traverse an angle less than 2π, which gives a meaning to the expression

$$|\alpha_1 - \alpha_2| \in [0, 2\pi)$$

as shown in Figure 2. Evidently, we have

$$|\alpha_2 - \alpha_1| = 2\pi - |\alpha_1 - \alpha_2|.$$

Definition 1. Let p be a point in the Riemann surface X. A ***sector at*** p is an open subset $\mathsf{A} \subset \mathsf{S}_p\mathsf{X}$ bounded by two distinct rays. We denote by (α_1, α_2) the sector that starts from α_1 and goes around the circle counter-clockwise to α_2. The ***angle of a sector*** $\mathsf{A} = (\alpha_1, \alpha_2)$ is the angle

$$|\mathsf{A}| = |\alpha_2 - \alpha_1| \in (0, 2\pi)$$

determined by its endpoints.

2.2 Real oriented blowup

We now take a point $p \in \mathsf{X}$ in our Riemann surface. The ***real oriented blowup of*** X ***at*** p is the space obtained by deleting p and replacing it with the set of rays through p:

$$[\mathsf{X} : p] = (\mathsf{X} \setminus \{p\}) \coprod \mathsf{S}_p\mathsf{X}$$

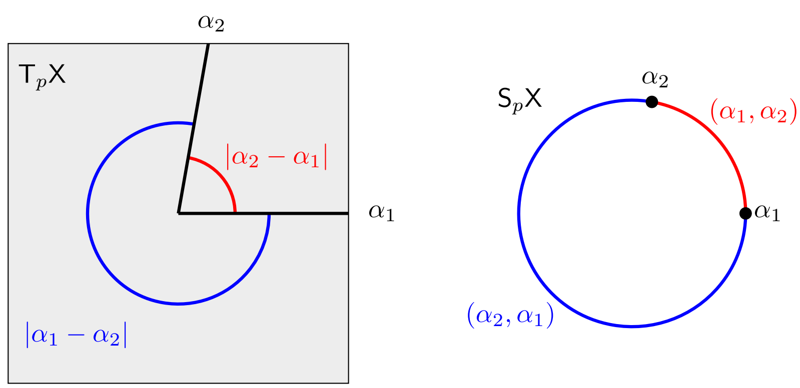

Figure 2: The angles formed by two rays in the tangent space $\mathsf{T}_p\mathsf{X}$, and the corresponding points and sectors in the circle $\mathsf{S}_p\mathsf{X}$

We claim that the set $[\mathsf{X} : p]$ naturally has the structure of a Riemann surface with boundary

$$\partial[\mathsf{X} : p] = \mathsf{S}_p\mathsf{X}$$

Thus points on the boundary correspond to tangent rays at p. See Figure 3 for an illustration. Indeed, if $x = re^{i\theta}$ is a local system of polar coordinates, then $[\mathsf{X} : p]$ is parametrized locally by the coordinates

$$(r, \theta) \in [0, \infty) \times (-\pi, \pi)$$

and the boundary is given by the locus $r = 0$. There is a natural map

$$[\mathsf{X} : p] \to \mathsf{X},$$

called the ***blowdown***, that is the identity away from p, and contracts $\mathsf{S}_p\mathsf{X}$ to p. It is given in coordinates by the obvious formula

$$(r, \theta) \mapsto re^{i\theta}.$$

With the real oriented blowup in hand, we may now make the

Definition 2. Let p be a point in the Riemann surface X. An open set

$$\mathsf{U} \subset \mathsf{X} \setminus \{p\} = [\mathsf{X} : p] \setminus \mathsf{S}_p\mathsf{X}$$

is a ***sectorial neighbourhood of*** p if it is connected, and its closure intersects the boundary $\mathsf{S}_p\mathsf{X}$ in an open sector $\mathsf{A} \subset \mathsf{S}_p\mathsf{X}$. This sector is then called the ***opening of*** U.

For example, Figure 4 shows a sectorial neighbourhood of the origin in complex line $\mathsf{X} = \mathbb{C}$.

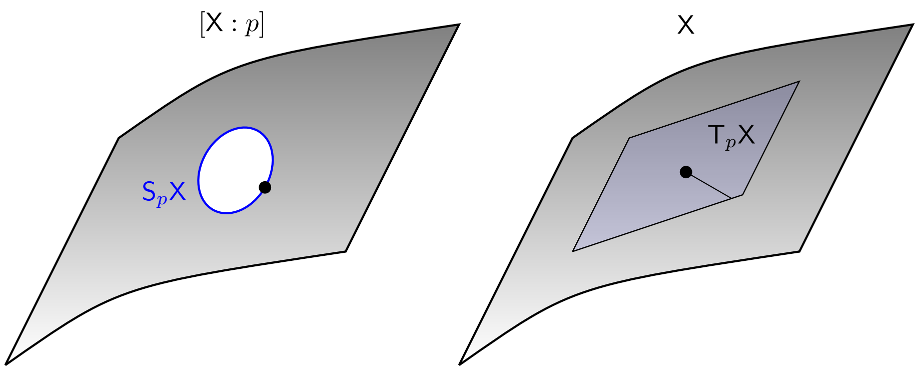

Figure 3: The real oriented blowup of the Riemann surface X at p, and a point on the boundary corresponding to a ray in the tangent space at p.

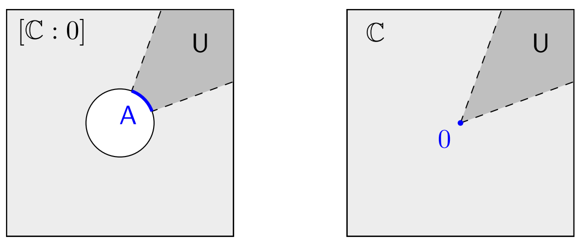

Figure 4: An open sectorial neighbourhood U of $0 \in \mathbb{C}$ with opening A.

3 Functions with exponential growth

3.1 Basic definitions

For the rest of the lecture, we will be concerned with properties of holomorphic functions that are defined in sectorial neighbourhoods of a point, starting with functions that have exponential growth. We will make use of the standard notation from complex geometry: if $\mathsf{U} \subset \mathsf{X}$ is an open set of a Riemann surface, then $\mathcal{O}(\mathsf{U})$ denotes the set of holomorphic functions on U.

Let $x = re^{i\theta}$ be a coordinate on X near p and suppose that U is a sectorial neighbourhood of p with opening $\mathsf{A} \subset \mathsf{S}_p\mathsf{X}$. We say that a holomorphic function $f \in \mathcal{O}(\mathsf{U})$ has ***exponential type in the direction*** $\alpha \in \mathsf{A}$ if we can find constants $\tau \in \mathbb{R}$ and $C > 0$, and a sectorial subneighbourhood $\mathsf{U}' \subset \mathsf{U}$ whose opening contains α, such that

$$|f| \le Ce^{\tau/r}$$

on U'. Thus the growth of the function f is at most exponential as we approach the point p in the direction α.

Exercise 1. Show that this condition depends only on the function f and the direction α, not on the coordinate x. $\square$

In order to measure the growth rate of f in the direction α, we consider the quantity

$$\tau(\alpha) = \inf_{C>0, \overline{\mathsf{U}'} \ni \alpha} \left\{ \tau \in \mathbb{R} \,\middle|\, |f| \leq C e^{\tau/r} \text{ on } \mathsf{U}' \right\}$$

Thus $\tau(\alpha)$ is the smallest exponent required to bound f as we approach p in the direction α. We then say that f ***has exponential type***

$$\frac{\tau(\alpha)}{r}$$

in the direction α. We say that f ***decays exponentially in the direction*** α if $\tau(\alpha) < 0$. The exponential type depends continuously on the direction α, and has a coordinate-invariant meaning on the boundary circle $\mathsf{S}_p\mathsf{X}$ of the blowup, as follows:

Exercise 2. Using polar coordinates $x = re^{i\theta}$ on the blowup $[\mathsf{X} : p]$, let ∂_r be the outward-pointing radial vector field. Its restriction to the boundary gives a section of the normal bundle of the boundary:

$$[\partial_r|_{\mathsf{S}_p\mathsf{X}}] \in \Gamma(\mathsf{S}_p\mathsf{X}, \mathsf{N}_{\mathsf{S}_p\mathsf{X}}).$$

By replacing the symbol $\frac{1}{r}$ above with this normal vector field, show that the exponential type of a function may be naturally viewed as a section of the normal bundle. $\square$

More generally, we could have functions of exponential type τ/r^β for $\beta > 0$, which are bounded by exponentials of the form

$$|f| \leq Ce^{\tau/r^\beta}$$

Their exponential types can be interpreted as sections of the bundle $(\mathsf{N}_{\mathsf{S}_p\mathsf{X}})^\beta$ of βth densities. The basic example is as follows:

Example 1. Let $\mathsf{X} = \mathbb{C}$ with the standard coordinate $x = re^{i\theta}$. And let U be a sectorial neighbourhood of the origin. Let $b, \beta > 0$ be constants and consider the function

$$\exp(-b/x^\beta),$$

defined on U using a choice of branch of x^β. We have

$$\mathrm{Re}(x^{-\beta}) = r^{-\beta}\cos(\theta\beta)$$

so that

$$|\exp(-b/x^\beta)| = \exp(-b\cos(\theta\beta)/r^\beta).$$

Evidently this function has exponential type

$$\frac{-b\cos(\beta\theta)}{r^\beta} \in \Gamma(\mathsf{S}_p\mathsf{X}, \mathsf{N}^\beta_{\mathsf{S}_p\mathsf{X}})$$

It decays exponentially in the region where $\cos(\beta\theta)$ is positive, and grows exponentially where $\cos(\beta\theta)$ is negative. Figure 5 shows the exponential type of the functions $\exp(1/x^k)$ for small integer values of k.

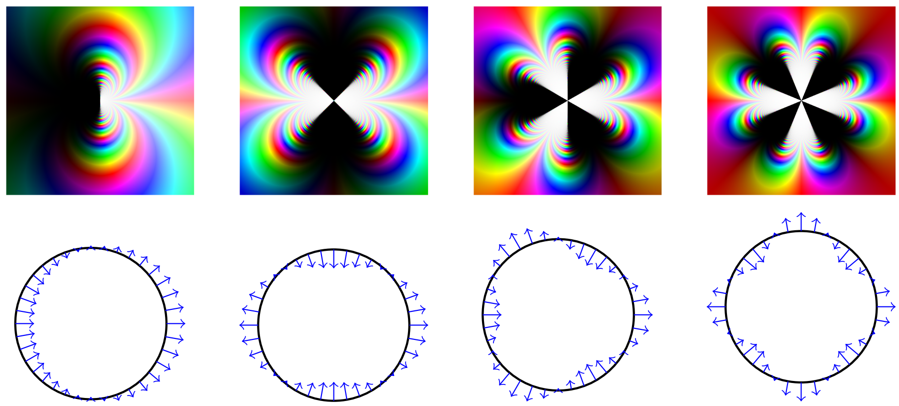

Figure 5: Contour plots of the functions e^{1/x^k} for $1 \leq k \leq 3$ and the corresponding exponential types. The latter are given by $\cos(k\theta)/r^k$; the arrows point outwards in the regions where the function is blowing up and inwards where it is decaying.

In this course we, will mainly deal with the case in which the exponential growth is like $\exp(\tau/r)$, therefore:

> **Unless we explicitly declare otherwise, we will assume that "exponential type" means that the growth is like $\exp(\tau/r)$, i.e. $\beta = 1$.**

3.2 The Fragmen–Lindelöf principle

Notice an important fact about the function $\exp(\tau/x)$: the regions in which it blows up or decays exponentially are sectorial neighbourhoods of angle π. In particular, the function cannot decay in any sector of opening bigger than π. That this observation holds true for any function of exponential type is the content of the following Fragmen–Lindelöf principle, which we will state momentarily.

The proof of this principle is a consequence of the classical ***maximum principle***: if f is a holomorphic function on a bounded region U that extends continuously to the boundary $\partial\mathsf{U}$, then

$$\sup_{\mathsf{U}} |f| \leq \sup_{\partial\mathsf{U}} |f|.$$

The other half of the Fragmen–Lindelöf principle, For functions of exponential type, a similar statement holds within sufficiently small sectorial neighbourhoods:

Theorem 1 (Fragmen–Lindelöf principle)**.** *Let X be a Riemann surface and let p be a point in X. Let U be a bounded sectorial neighbourhood of p with opening*

A, *and let* $f \in \mathcal{O}(\mathsf{U})$ *be a holomorphic function. Then the following statements hold*

1. *(Maximum principle) Suppose that* $|\mathsf{A}| < \pi$*, that* f *is of exponential type, and that* f *extends continuously to* $\partial\mathsf{U} \setminus \mathsf{A}$*. Then*

$$\sup_{x\in\mathsf{U}} |f(x)| \leq \sup_{\partial\mathsf{U}\setminus\mathsf{A}} |f|$$

2. *If* $|\mathsf{A}| > \pi$ *and* f *decays exponentially in all directions of* A*, then* $f = 0$*.*

Proof. We follow [1, p. 134–135]. It is evidently sufficient to prove the statements when U lies in the domain of a coordinate $x = re^{i\theta}$ centred at p. We may further assume that A is centred along the positive real axis in this coordinate. The strategy for both statements is to apply the classical maximum principle to a family of functions that limits to our given function f, and whose modulus is controlled.

For the first statement, choose constants $\beta, b > 0$ such that

$$1 < \beta < \pi/|\mathsf{A}|.$$

This is possible because $|\mathsf{A}| < \pi$. Then the function $\exp(-b/x^\beta)$ has exponential type

$$\frac{-b\cos(\beta\theta)}{r^\beta}$$

and $\cos(\beta\theta)$ is positive on all of A, so that the function decays. Now f is of exponential type $\tau(\theta)/r$ for some τ. Therefore $fe^{-b/x^\beta} \to 0$ as $r \to 0$ because $r^{-\beta}$ blows up more rapidly than r^{-1}. So even though f may not extend continuously to the sector $\mathsf{A} \subset \partial\mathsf{U}$, the function fe^{-b/x^β} does extend; its value on A is zero. The result now follows by applying the classical maximum principle to the function fe^{-b/x^β} and sending $b \to 0^+$.

For the second statement, notice that we may assume that U has the form

$$\mathsf{U} = \left\{ re^{i\theta} \,\middle|\, r \in (0, R) \text{ and } \theta \in \left(-\tfrac{|\mathsf{A}|}{2}, \tfrac{|\mathsf{A}|}{2}\right) \right\},$$

i.e. that it is a sector bounded by straight rays with radius $R > 0$ and opening A. Indeed, any other sectorial neighbourhood of p with opening at least π contains a subneighbourhood of this form, and by analytic continuation, it is enough to show that the function vanishes on such a subneighbourhood.

We set

$$\beta = \frac{\pi}{|\mathsf{A}|} < 1$$

and consider the function $\exp(\lambda/x^\beta)$ for $\lambda > 0$, which has exponential type

$$\frac{\lambda\cos(\beta\theta)}{r^\beta}$$

Now the boundary of U consists of the sector A, the two rays of angles $\theta = \pm\frac{|\mathsf{A}|}{2}$ and length R, and the arc of fixed radius R between these two rays. By construction, the function $\exp(\lambda/x^\beta)$ has unit modulus along the two rays. Moreover, since $\beta < 1$, and f decays exponentially, the function fe^{λ/x^β} decays to 0 on A. The result now follows easily by applying the classical maximum principle to fe^{λ/x^β} and taking $\lambda \to \infty$. $\square$

4 Asymptotic expansions

4.1 Poincaré's definition and its basic properties

Suppose that U is a sectorial neighbourhood of p with opening $\mathsf{A} \subset \mathsf{S}_p\mathsf{X}$, and that x is a coordinate at p whose domain contains U. Suppose that $f \in \mathcal{O}(\mathsf{U})$ is a holomorphic function defined in this region.

Following Poincaré, we say that f is ***asymptotic to the power series*** $\sum_{k=0}^{\infty} a_k x^k$ ***in*** U, and write

$$f \sim \sum_{k=0}^{\infty} a_k x^k$$

if for every sectorial subneighbourhood $\mathsf{U}' \subset \mathsf{U}$ whose opening is strictly smaller than A, and for every $N \in \mathbb{Z}_{>0}$ there exists a constant $C > 0$ such that

$$\left| f(x) - \sum_{n=0}^{N} a_n x^n \right| \le C|x|^{N+1}.$$

In other words, the series is a good approximation to the function near p. For any $N \in \mathbb{Z}_{>0}$, we may evidently write

$$f(x) = \sum_{n=0}^{N} a_k x^k + x^{N+1}\phi(x)$$

where ϕ is holomorphic in U and bounded near the opening A. Notice, in particular, f has a limit as we approach the boundary sector A:

$$\lim_{x\to 0} f(x) = a_0$$

In other words, from the point of view of this sector, the function has a well-defined value at p. We will see shortly that an asymptotic expansion, if it exists, is unique. However, the expansion does not uniquely determine the function.

Definition 3. We denote by $\widetilde{\mathcal{O}}(\mathsf{U}) \subset \mathcal{O}(\mathsf{U})$ the functions that admit an asymptotic expansion in the sectorial neighbourhood U of p.

Example 2. If f is a holomorphic function defined in an actual neighbourhood p (rather than a sectorial one), then f is asymptotic to its Taylor series. $\square$

Example 3. Let U be the right half-plane $\operatorname{Re}(x) > 0$, viewed as a sectorial neighbourhood of the origin. Consider the function $f = e^{-1/x}$ defined in this region. Then f decays exponentially in this sector, and hence it vanishes more rapidly than any polynomial as $x \to 0$ in U. It follows that

$$f = e^{-1/x} \sim 0$$

in U. Hence f and 0 have the same expansion. But outside of U, the function f admits no asymptotic power series expansion whatsoever. Indeed, for $\operatorname{Re}(x) < 0$ the function blows up and so it has no limit as $x \to 0$. Meanwhile, along the imaginary axis, this function is oscillating wildly, so it doesn't have a limit as $x \to 0$ along the imaginary axis either. □

We now establish some basic properties of asymptotic expansions:

Exercise 3. Show that the set $\widetilde{\mathcal{O}}(\mathsf{U}) \subset \mathcal{O}(\mathsf{U})$ of holomorphic functions that admit asymptotic expansions is closed under linear combinations and products, i.e. it is a subalgebra of $\mathcal{O}(\mathsf{U})$. □

Exercise 4. Show that the existence of an asymptotic expansion of f is independent of the chosen coordinate. □

Proposition 1. *If $f \in \widetilde{\mathcal{O}}(\mathsf{U})$ has an asymptotic expansion*

$$f \sim \sum_{k=0}^{\infty} a_k x^k$$

then its differential $df = f'(x)dx$ also has an asymptotic expansion

$$df \sim \left(\sum_{k=1}^{\infty} k a_k x^{k-1} \right) dx.$$

Proof. We follow [1, p. 130]. Suppose we are given a proper subneighbourhood $\mathsf{U}' \subset \mathsf{U}$. We need to establish the bound

$$\left| f'(x) - \sum_{k=1}^{N} k a_k x^{k-1} \right| < C x^N$$

as $x \to 0$ in U'. For this we write

$$f(x) - \sum_{k=1}^{N} a_k x^k = x^{N+1} \phi(x)$$

where ϕ is a bounded holomorphic function. We compute

$$f'(x) - \sum_{k=1}^{N} k a_k x^{k-1} = x^N ((N+1)\phi + x\phi')$$

Since we know that ϕ is bounded, it is enough to show that $x\phi'$ is also bounded. But Cauchy's formula gives

$$\phi'(x) = \frac{1}{2\pi} \int_{\gamma_x} \frac{\phi(y)dy}{(y-x)^2}$$

where γ_x is a circle of arbitrary radius $R(x)$ centred about x, and hence

$$|x||\phi'(x)| \leq |x| \cdot \frac{1}{2\pi}|\gamma_x| \cdot \frac{\sup_{\gamma_x}|\phi|}{R(x)^2} = \frac{|x|}{R(x)} \sup_{\gamma_x} |\phi|.$$

We leave it to the reader to verify that the circles can be chosen so that, as $x \to 0$, the radius $R(x)$ decreases linearly and $\sup_{\gamma_x}|\phi|$ is globally bounded. Thus $|x\phi'|$ is bounded and the result follows. □

Remark 1. At this point, we must issue an important warning: differentiation of asymptotic series fails if we work with functions on $\mathbb{R}$ instead of holomorphic functions in open sectors. (Notice that the proof relied heavily on our ability to draw circles in the complex plane, so that we could use the Cauchy formula.)

For example, consider the function

$$f = e^{-1/x}\cos(e^{1/x})$$

on the real line $x \in \mathbb{R}$. Since cos is bounded and $e^{-1/x}$ decays rapidly, this function is asymptotic to 0 as $x \to 0^+$. Differentiating, we find

$$f' = x^{-2}e^{-1/x}\cos(e^{1/x}) - x^{-2}\cos(e^{1/x}).$$

Now, the first term remain asymptotic to zero, but the second term blows up in magnitude and oscillates wildly as $x \to 0$, so the function cannot be asymptotic to any power series. The reason for the problem is best seen by returning to the complex plane; a contour plot is shown in Figure 6. The function behaves in a very complicated manner on either side of the real axis, so we are unable to produce an asymptotic expansion in any sectorial neighbourhood of the axis. □

4.2 Existence of asymptotic expansions

Now that we know that it is possible to take derivatives of asymptotic expansions, it is straightforward to determine which functions admit asymptotic expansions:

Proposition 2. *Let* U *be a sectorial neighbourhood of a point* p *in a Riemann surface. A function* $f \in \mathcal{O}(\mathsf{U})$ *admits an expansion in* U *if and only if all of its derivatives are bounded in any proper subsector* $\mathsf{U}' \subset \mathsf{U}$*. In this case, we have*

$$a_n = \lim_{x \to 0} \frac{1}{n!} f^{(n)}(x).$$

Proof. Similar to Taylor's theorem; see e.g. [1, p. 130]. □

Figure 6: A contour plot of the function $e^{-1/x}\cos(e^{1/x})$.

Corollary 1. *The asymptotic expansion of a given function f, if it exists, is unique.*

What this means is that there is a canonically defined map

$$\widetilde{\mathcal{O}}(\mathsf{U}) \to \mathbb{C}[[x]],$$

which extracts the asymptotic expansion of a function with respect to the coordinate x. The kernel of this map consists of all functions that vanish faster than any polynomial as $x \to 0$.

Because of the expression for the coefficients in Proposition 2, this procedure behaves like a Taylor expansion under changes of coordinate. It may therefore be viewed more invariantly as a map

$$\widetilde{\mathcal{O}}(\mathsf{U}) \to \widehat{\mathcal{O}}_{\mathsf{X},p}$$

where $\mathcal{O}_{\mathsf{X},p}$ is the formal completion of the structure sheaf of X at p.

Evidently, the expansion depends only on the germ of a function near the opening $\mathsf{A} \subset \partial\mathsf{U}$. Hence for any sector, we have a map

$$\widetilde{\mathcal{O}}(\mathsf{A}) \to \widehat{\mathcal{O}}_{\mathsf{X},p}$$

where $\widetilde{\mathcal{O}}(\mathsf{A})$ is the set of germs of functions that admit asymptotic expansions in a sectorial neighbourhood of p with opening A.

We can therefore state the problem of resummation of divergent series as follows:

> **Given a formal power series $\hat{f}$ and a sector A, find a function that has $\hat{f}$ as its asymptotic expansion along A.**

In posing this problem, it is important that we work in sectors, which have opening angles less than 2π by definition. If we instead try to solve the problem for a function defined in a whole punctured disk around p, we encounter an obstruction. Indeed, any function that is defined on a punctured disk at p, and which admits an asymptotic expansion at p, is necessarily bounded on the whole disk. But the Riemann removable singularity theorem implies that such a function is holomorphic at p, and so its asymptotic expansion is simply its Taylor expansion, which converges. Thus it is impossible to resum any divergent series on a whole disk. However, it is always possible find sums in sectors:

Proposition 3. *Every formal power series may be realized as the asymptotic expansion of a function defined in a sectorial neighbourhood of a point* $p \in \mathsf{X}$. *Moreover, the opening* $\mathsf{A} \subset \mathsf{S}_p\mathsf{X}$ *of such a neighbourhood may be chosen arbitrarily. Put differently, the expansion map gives an exact sequence*

$$0 \longrightarrow \widetilde{\mathcal{O}}_0(\mathsf{A}) \longrightarrow \widetilde{\mathcal{O}}(\mathsf{A}) \longrightarrow \widehat{\mathcal{O}}_{\mathsf{X},p} \longrightarrow 0$$

for all sectors A, *where* $\widetilde{\mathcal{O}}_0(\mathsf{A})$ *denotes the functions that vanish faster than any polynomial.*

Proof. Choose a coordinate $x = re^{i\theta}$ near p. Without loss of generality, we may assume that x is defined in a disk of radius 1 and that A is the sector $(-\theta_0, \theta_0)$ bisected by the real axis. Therefore $|\mathsf{A}| = 2\theta_0$.

Suppose given a series

$$\hat{f} = \sum_{k=0}^{\infty} a_k x^k$$

We wish to find a function f that is asymptotic to f near 0. The series, will not, in general, converge. But we claim that it is possible to construct a convergent series by adding exponentially small corrections:

$$f = \sum_{k=0}^{\infty} a_k x^k (1 - e^{-b_k/x^\beta}).$$

This series will then define a holomorphic function that has $\hat{f}$ as its asymptotic series, since the exponentially small corrections do not alter its asymptotic expansion.

In order to see that such a series exists, we choose the constant β so that $\beta|\mathsf{A}| < \pi$. Then the function e^{-b/x^β} has exponential type

$$\frac{-b\cos(\beta\theta)}{r^\beta},$$

so that it decays in all directions of A, provided that $b > 0$. Using Taylor's theorem we obtain for any $\epsilon > 0$ an estimate of the form

$$|1 - e^z| \leq C(\epsilon)|z|,$$

for $|\arg(z)| < \frac{\pi}{2} - \epsilon$ and $C(\epsilon) > 0$. We therefore have

$$\left| a_k x^k (1 - e^{-b_k/x^\beta}) \right| \leq C \frac{b_k |a_k| |x|^k}{|x|^\beta}$$

Hence the series will converge so long as we choose the coefficients b_k to decay sufficiently rapidly as $k \to \infty$. □

Now that we know that all power series are asymptotic expansions, the problem is that there will be many functions that have a given asymptotic expansion. So a better way of stating the problem is as follows:

> **Given a formal power series $\hat{f}$ and a sector A, find a function that has $\hat{f}$ as its asymptotic expansion along A, and is *uniquely specified* by some additional condition**.

For example, we might want our resummation procedure to be compatible with additional operations, such as multiplication and differentiation of functions. In algebraic terms, we wish to find a natural splitting of the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{\mathcal{O}}_0(\mathsf{A}) \arrow[r] & \widetilde{\mathcal{O}}(\mathsf{A}) \arrow[r] & \widehat{\mathcal{O}}_{\mathsf{X},p} \arrow[r] \arrow[l, dashed, bend right] & 0 \end{tikzcd}$$

at least for some sufficiently nice subalgebra of $\widehat{\mathcal{O}}_{\mathsf{X},p}$. This is the goal that we will be working towards in the coming lectures.

References

[1] B. Y. Sternin and V. E. Shatalov, *Borel-Laplace transform and asymptotic theory*, CRC Press, Boca Raton, FL, 1996. Introduction to resurgent analysis.