Resurgence in Geometry and Physics

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Lecture 1

Abstract

This course is about the theory of "resurgence", which is a method for dealing with divergent series expansions that arise in various parts of mathematics and physics. In this introductory lecture, we give some basic motivating examples and outline our plans for the course.

Contents

1 What the course is about

This course is about fairly elementary mathematical objects, namely series which have a form something like

$$\sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty} a_{jk}x^k\right)e^{-c_j/x},$$

and define functions of the variable x. Here a_{jk} and c_k for $j,k \in \mathbb{Z}_{\geq 0}$ are constants, either real or complex. We think of these series as Taylor series that have been enhanced to include exponentially small corrections: if $c_j > 0$, then $e^{-c_j/x}$ decays very rapidly as x approaches 0 along the positive real line. Indeed, the decay is faster than any polynomial, and so there is no way that such terms could be detected by the Taylor expansion of an analytic function; they produce essential singularities. But the real issue is that even the power series part will almost always diverge.

There are many examples of such series "in nature". For example, the series could represent the solution of an ordinary differential equation for a function of x, or the value of some integral in which x appears as a parameter. Perhaps the series gives the value of a physical quantity of interest, such as the energy, by expanding it in a small parameter, such as the electron charge. Many geometers and physicists have recently become interested in these series due to their appearance (at least implicitly) in numerous topics at the forefront of research, including:

- Normal forms for dynamical systems
- Gauge theory of singular connections
- Quantization of symplectic and Poisson manifolds
- Floer homology and Fukaya categories
- Knot invariants
- Wall-crossing and stability conditions in algebraic geometry
- Spectral networks
- WKB approximation in quantum mechanics
- Perturbative expansions in quantum field theory (QFT)

This last example is, in some sense, the ultimate one, as most of the others can be interpreted as calculations in some low-dimensional QFT. At the same time, it presents the greatest mathematical difficulties, since QFT does not, in general, have a rigorous mathematical definition. But there has been some recent work in the physics literature suggesting the possibility that the divergent series obtained from the perturbative expansion may have more information about the true nature of the QFT that one might naively expect. These ideas are the main motivation for this course, although it is unlikely that we will discuss most of them in any detail.

Indeed, the bulk of the course will simply be concerned with the basic theory of these series. One approach to the subject, which is known as "resurgence", was developed by Écalle in the 1970s, although it builds on earlier work of many others, going back at least to Émile Borel at the turn of the 20th century. One might reasonably expect it to be a straightforward extension of classical real or complex analysis, and at a basic level, this expectation is correct. But we will quickly find that the devil is in the details. A number of remarkable objects and subtle phenomena will appear, including infinite-sheeted Riemann surfaces, wall-crossing behaviour, and a new operator called the "alien derivative". The theory has as much an algebraic and geometric character as an analytic one.

For a hint of where the subtlety lies, we begin our discussion with the following simple example.

2 Solutions of ordinary differential equations

2.1 Euler's equation

Consider the following ordinary differential equation (ODE), sometimes known as ***Euler's equation***:

$$x^2 \frac{df}{dx} = x - f \tag{1}$$

Our aim is to find all possible solutions of this equation on the real line $x \in \mathbb{R}$.

The equation can be solved quite readily using the method of integrating factors, but let us pretend that we are not so clever, and try to solve it by brute force. We therefore search for a solution in the form of a power series

$$f = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$$

Substituting this series into (1), we obtain

$$\begin{aligned} x^2(a_1 + 2a_2 x + 3a_3 x^2 + \cdots) &= x - (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots) \\ a_1 x^2 + 2a_2 x^3 + 3a_3 x^4 + \cdots &= -a_0 - (a_1 - 1)x - a_2 x^2 - a_3 x^3 - a_4 x^4 + \cdots \end{aligned}$$

Comparing the coefficients of the two sides of the equation, we find

$$\begin{aligned} 0 &= -a_0, \\ 0 &= -(a_1 - 1) \\ (k-1)a_{k-1} &= -a_k \quad \text{for } k \geq 2. \end{aligned}$$

From this recursion, it is clear that

$$a_k = (-1)^{k+1}(k-1)!$$

for $k \geq 1$, and therefore

$$f = \sum_{k=0}^{\infty} (-1)^k k! x^{k+1}. \tag{2}$$

This answer is nice and simple, but it presents a couple of major problems:

1. Applying the ratio test, we compute

   $$\lim_{k\to\infty} \frac{|(k+1)! x^{k+2}|}{|k! x^{k+1}|} = \lim_{k\to\infty} k|x| = \infty$$

   for any nonzero value of x. Hence the series diverges, or more precisely, its radius of convergence is zero, which is rather unsettling.

2. We know that solutions of first-order differential equations are not unique, but should instead depend on a parameter, corresponding to the choice of an initial condition, say at $x = 0$. But here we found only a single solution.

   To find the other solutions, we observe that (1) is a linear ODE, but it is inhomogeneous. The associated homogeneous equation

$$x^2 \frac{dg}{dx} = -g$$

   is easily seen to have solutions

$$g = ae^{1/x}$$

   where $a \in \mathbb{C}$ is an arbitrary constant. Hence the general solution of (1) must have the form

$$f = \sum_{k=0}^{\infty} (-1)^k k! x^{k+1} + ae^{1/x} \tag{3}$$

   This is a very simple example of the general form of series that we will be discussing.

Suppose that we pretend, for the moment, that our divergent series was actually the convergent Taylor expansion of an analytic function. Then we could quite easily plot the solution curves in a neighbourhood of $x = 0$. Indeed, if we set the arbitrary constant a to be zero, we could plug in $x = 0$ and find that $f(0) = 0$, $f'(0) = 1$, etc. So there is a solution which passes through the origin with slope 1. On the other hand the function $e^{1/x}$ blows up to ∞ very rapidly as $x \to 0^+$, and decays very rapidly as $x \to 0^-$. The general solution is then obtained by adding these two observations together, resulting in Figure 1.

These two "wrongs" arose because at the point $x = 0$, where we have centred our expansion, the equation (1) has a *singularity*. This becomes more clear if we divide through by x^2, so that the equation is rewritten in the form

$$\frac{df}{dx} = x^{-1} - x^{-2} f.$$

In this form, it is evident that the equation has a pole when $x = 0$, and so the usual existence and uniqueness theorem for analytic solutions of the equation breaks down there. Indeed, the exponential correction $e^{1/x}$ that we missed has an essential singularity at $x = 0$, so there is no way we could have found it by power series.

The presence of the singularity suggests that we should instead try to expand about a different point $x_0 \neq 0$, where the equation is analytic. Then the problems above would not appear, but the series would unfortunately not have nearly so simply an expression. More importantly, such expansions would not help us understand the most interesting aspect of the equation, which is the

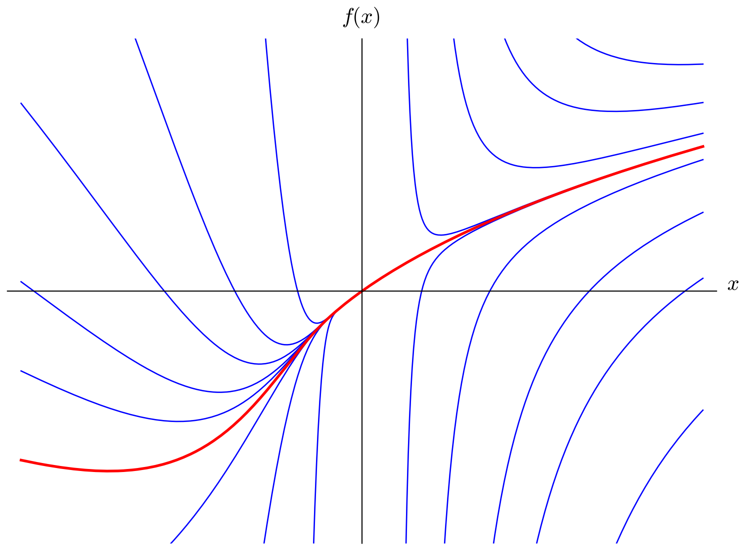

Figure 1: The solutions of Euler's ODE in the neighbourhood of $x = 0$. The red curve is the "function" $\sum_{k=0}^{\infty}(-1)^k k! x^{k+1}$ and the other solutions are obtained by adding a term $ae^{1/x}$.

interesting behaviour of the solutions near the singular point $x = 0$ that we have guessed using our series. So expanding about x_0 is not a very satisfying solution to the problem.

We thus find ourselves wishing that there was a way to salvage our elegant series solution (2)—i.e. to somehow "resum" the divergent series and obtain a solution of the equation, thus verifying the qualitative properties of the solutions that we have guessed. Remarkably, just by resumming this single divergent series, we will in fact recover the other term $ae^{1/x}$ that we thought were missing.

2.2 Summing the series

There are many approaches to resumming divergent series, such as Abel summation, Césaro summation, ζ-function regularization, etc. The method that is appropriate for our setting was introduced by Émile Borel in 1899, and is therefore called ***Borel summation***. It can can be defined in various ways, but there is a nice non-rigorous heuristic based on the following identity, which is easily obtained using integration by parts:

$$x^{k+1} = \frac{1}{k!}\int_0^\infty t^k e^{-t/x} dt. \tag{4}$$

The reader may recognize the integral as the Laplace transform of the function t^k, written in the variable $s = x^{-1}$.

Let us substitute the identity (4) into our series solution

$$f = \sum_{k=0}^{\infty} (-1)^k k! x^{k+1}.$$

We compute

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} (-1)^k k! x^{k+1} \\ &= \sum_{k=0}^{\infty} k! \left(\frac{(-1)^k}{k!} \int_0^{\infty} t^k e^{-t/x} dt \right) \\ &= \int_0^{\infty} \left(\sum_{k=0}^{\infty} (-1)^k t^k \right) e^{-t/x} dt \\ &= \int_0^{\infty} \frac{1}{1+t} e^{-t/x} dt. \end{aligned}$$

Of course, this calculation was illegal for several reasons. First of all, the theorems that allow us to interchange the order of summation and integration rest on the assumption that the series converges. This difficulty is compounded by the fact that the integral is over an unbounded interval. Secondly, the formula

$$\sum_{k=0}^{\infty} (-1)^k t^k = \frac{1}{1+t}$$

is only valid when $|t| < 1$; this problem is less serious, because the theory of analytic continuation implies that there is really only one natural way to extend the function to the region $|t| \geq 1$.

On the other hand, let us observe that if x is a positive real number, then the integrand

$$\frac{e^{-t/x}}{1+t}$$

decays very rapidly as $t \to \infty$. Hence the expression

$$f(x) = \int_0^{\infty} \frac{e^{-t/x}}{1+t} dt \tag{5}$$

actually defines a nice smooth function of x, at least if x is positive. This function is the ***Borel sum*** of our divergent series.

Although the Borel sum was obtained in a rather ad hoc fashion, it is, in fact, the function we are looking for. Indeed, differentiating under the integral sign, we find

$$
\begin{aligned}
\frac{df}{dx} &= \int_0^\infty \frac{d}{dx}\left(\frac{e^{-t/x}}{1+t}\right) dt \\
&= \int_0^\infty \frac{e^{-t/x}}{1+t} \cdot \frac{t}{x^2} dt \\
&= \frac{1}{x^2}\int_0^\infty e^{-t/x} \frac{t}{1+t} dt \\
&= \frac{1}{x^2}\int_0^\infty e^{-t/x} \left(1 - \frac{1}{1+t}\right) dt \\
&= \frac{1}{x^2}\int_0^\infty e^{-t/x} dt - \frac{1}{x}\int_0^\infty \frac{e^{-t/x}}{1+t} dt \\
&= x^{-1} - x^{-2} f
\end{aligned}
$$

Unlike our previous calculation, this one is completely rigorous, and it shows us that the function (5) is actually a solution of our original ODE for $x > 0$. In fact, with a bit more work, one can check that f has infinitely many right-handed derivatives as $x \to 0^+$, and that our divergent series is, in fact, its Taylor expansion. Thus, even though the series diverges, Taylor's theorem ensures that the partial sums give a good approximation to the function for small values of x. Indeed, we have

$$
\left| f(x) - \sum_{k=0}^{n-1} (-1)^k k! x^{k+1} \right| < n! |x|^{n+1}
$$

for $x > 0$. Thus the series is what's known as an ***asymptotic expansion*** of the function f.

2.3 Integration contours and the Stokes phenomenon

The expression

$$
f(x) = \int_0^\infty \frac{e^{-t/x}}{1+t} dt
$$

gives us a sum for our divergent series, provided that $x > 0$. But what about the solutions for $x < 0$? In this case, the integral evidently diverges, so the formula is not valid.

At this point it is helpful to expand our viewpoint, and allow x to be complex-valued, rather than real-valued. We observe that, for the integral to converge, all that is really necessary is that $\mathrm{Re}(x) > 0$, so this expression makes sense for any value of x in the right half-plane, giving a complex analytic function. So we should try to use analytic continuation to define the function in the region where $\mathrm{Re}(x) < 0$.

To do so, it is useful to modify our contour of integration, as illustrated in Figure 2. Indeed, if γ_x denotes the ray from 0 to ∞ in the direction of x, then

$$f(x) = \int_{\gamma_x} \frac{e^{-t/x}}{1+t} dt$$

will converge: indeed, along γ_x, the ratio $-t/x$ is real and negative, so that $e^{-t/x}$ decays very rapidly as $t \to \infty$ in this direction. As long as x does not lie on the negative real axis $\mathbb{R}_{<0} \subset \mathbb{C}$, the contour γ_x will avoid the pole of the integrand, and the integral will converge to give us a perfectly good value for $f(x)$. Thanks to the path-independence of contour integrals, this new definition of $f(x)$ is the same as the previous one whenever $\mathrm{Re}(x) > 0$, and so we have obtained an analytic continuation of the solution $f(x)$ to all values of $x \in \mathbb{C} \setminus \mathbb{R}_{<0}$.

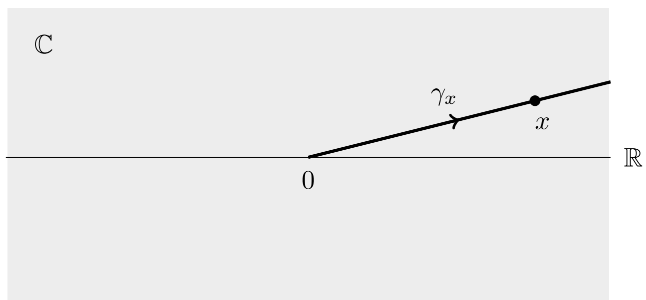

Figure 2: Rotating the integration contour in the t-plane

In fact, we cannot do any better, as the negative real axis $\mathbb{R}_{<0}$ is a branch cut for the function f. To see why, let us pick a point $x \in \mathbb{R}_{<0}$. We can't compute the value of $f(x)$ by integration along the corresponding ray, because that ray passes through the pole of the integrand at $t = -1$. But the situation can be remedied by deforming the contour a wee bit, in order to avoid the pole. As shown in Figure 3, we end up with two possible contours Γ_+ and Γ_-, depending on whether we pass above or below the pole, and hence two possible values

$$f(x)^+ = \int_{\Gamma_+} \frac{e^{-t/x}}{1+t} dt \qquad\qquad f(x)^- = \int_{\Gamma_-} \frac{e^{-t/x}}{1+t} dt.$$

Clearly, these are the values of $f(x)$ that are obtained by analytic continuation through the upper and lower half-planes, respectively.

To see that the solution is indeed multivalued, we compute

$$\begin{aligned} f(x)^+ - f(x)^- &= \int_{\Gamma_+ - \Gamma_-} \frac{e^{-t/x}}{1+t} dt \\ &= 2\pi i \cdot \mathrm{Res}_{t=-1} \left(\frac{e^{-t/x}}{1+t} dt \right) \\ &= 2\pi i \cdot e^{1/x}. \end{aligned}$$

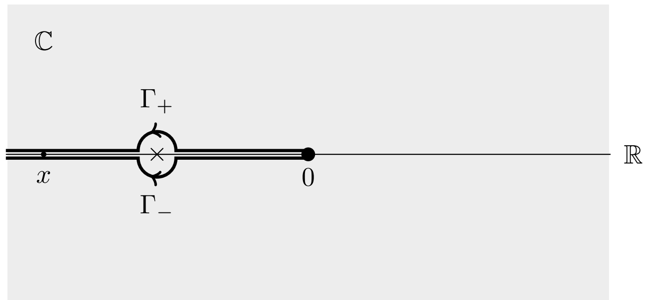

Figure 3: Avoiding the pole at $t = -1$

Thus $f(x)^+ - f(x)^{-1} \propto e^{1/x}$ is a nontrivial solution of the homogeneous equation (as the difference of any two distinct solutions must be). So by carefully analyzing the sum of the unique divergent series solution, we actually discover the other solutions that we thought were missing. Indeed, it is apparent that the general solution of the ODE for arbitrary x must have the form

$$f(x) = \int_{\gamma_x + \frac{a}{2\pi i}\Gamma_0} \frac{e^{-t/x}}{1+t} dt = \int_{\gamma_x} \frac{e^{-t/x}}{1+t} dt + ae^{1/x}$$

where Γ_0 is a positively oriented loop that wraps once around the singular point $t = -1$, and $a \in \mathbb{C}$ is a constant.

This behaviour, in which the function defined by a divergent series jumps along a ray, is an archetypal example of the **Stokes phenomenon**. It is the key to resurgence theory: the crucial missing term $e^{-1/x}$ has "resurged" from amidst the fog of the divergent series to produce the discontinuity.

3 Borel summation

Let us briefly summarize the Borel summation procedure that we have employed; we will be more precise about this process in the coming weeks. Starting from a power series in the variable x, say

$$f = \sum_{k=0}^{\infty} a_k x^{k+1},$$

we formed its ***Borel transform***

$$\hat{f} = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!},$$

which is a series in a new variable t. For $x > 0$, the ***Borel sum of*** f is given by

$$f(x) = \int_0^{\infty} \hat{f}(t) e^{-t/x} dt.$$

In order for this procedure to work, several things are required:

1. The Borel transform must converge, i.e. have a nonzero radius of convergence $R > 0$, so that it defines a function

$$\hat{f}_0(t) = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!}$$

   for $|t| < R$.

2. The function $f_0(t)$ so defined must extend to all values of $t > 0$, by setting

$$\hat{f}(t) = \text{ analytic continuation of } \hat{f}_0(t) \text{ for } t \in [0, \infty)$$

3. The function $\hat{f}(t)$ must not grow too quickly as $t \to \infty$, so that the integral

$$\int_0^{\infty} \hat{f}(t) e^{-t/x} dt$$

   will converge.

When these conditions hold, we say that the series f is ***Borel summable*** at $x \in \mathbb{R}_{>0}$. Similarly, we can discuss Borel summability for complex values of x by changing our contour of integration as above. It is easy to see that if the series f was already convergent, then this process would give back the usual sum of the series whenever x lies within its radius of convergence.

In general, there will be certain directions in which the Borel transform has singularities, such as branch points and poles. In most examples of physical interest, there will, in fact, be infinitely many such singularities. In these directions, the series is not Borel summable, at least as we have defined it here. But as we have seen, these singularities carry important information about the functions that are defined by the divergent series. The theory of resurgence is about understanding the singularities, and how they affect the summation process.

4 Bohr–Sommerfeld rules

Another source of divergent series with exponential corrections is quantum mechanics. We consider a particle with mass m moving on the real line $M = \mathbb{R}$ with coordinate x, subject to a potential $V : M \to \mathbb{R}$, which for us will simply be a polynomial in x. We will assume that $V(x) \to \infty$ as $x \to \pm\infty$, which means that the particle requires infinite energy to escape to infinity.

The classical energy of such a particle is given by the ***Hamiltonian function***

$$H = \frac{p^2}{2m} + V(x)$$

where p is the momentum. We view x and p as coordinates on the phase space $T^*M \cong \mathbb{R}^2$ (the cotangent bundle), so that the Hamiltonian is a function $H : T^*M \to \mathbb{R}$.

Evidently, this function can take on any continuous value greater than or equal to the minumum of $V(x)$. But in quantum mechanics, only certain discrete values are allowed. In general, to compute the correct energy levels, we are supposed to do a different calculation, namely to find the eigenvalues of the ***quantum Hamiltonian***, which is a differential operator

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d}{dx^2} + V(x),$$

acting on the smooth functions $C^\infty(M)$. Here $\hbar$ is Planck's constant, a universal constant that is independent of the particular system under consideration.

Finding the eigenvalues of a differential operator is a complicated problem, but there is a heuristic that can be used to obtain a first approximation using the classical Hamiltonian function. It is called the ***Bohr–Sommerfeld rule***, and it works as follows. Suppose we choose a value $E \in \mathbb{R}$ and would like to know whether it is a quantum-mechanically allowed value for the energy. Let $\gamma_E \subset T^*M$ be the locus where H is equal to E; in coordinates x and p, this locus is described by the polynomial equation

$$H(x,p) - E = 0,$$

and hence it is (generically) a smooth curve in the phase space T^*M. Define the ***action of*** γ_E to be the integral

$$S(E) = \int_{\gamma_E} p\,dx.$$

Then, to a first approximation, E is an allowed energy level if and only if

$$S(E) = 2\pi\hbar\left(n + \frac{1}{2}\right)$$

for some integer $n \in \mathbb{Z}$.

Example 1. Suppose that $V = \frac{1}{2}x^2$ is the potential of a simple harmonic oscillator. Then the level set γ_E is a circle of radius $\sqrt{2E}$, and hence it bounds the disk $D_E \subset T^*M$. Using Stokes' theorem, we have

$$\int_{\gamma_E} p\,dx = \int_{D_E} dp \wedge dx = -\pi(\sqrt{2E})^2 = -2\pi E.$$

Therefore we obtain the quantization condition

$$E = (n + \tfrac{1}{2})\hbar$$

for $n \in \mathbb{Z}_{\geq 0}$. Hence the successive energy levels are each separated by an amount equal to $\hbar$. These values are precisely the eigenvalues of the corresponding operator

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x),$$

so in this case the Bohr–Sommerfeld approximation is exact. $\square$

For most other potentials, the Bohr–Sommerfeld rule is not exact; it only gives an approximation. Instead, the equation

$$S(E) = 2\pi\hbar\left(n + \frac{1}{2}\right)$$

is really only the first order approximation of the correct condition, which has the form

$$S_\hbar(E) = S_0(E) + \hbar S_1(E) + \hbar^2 S_2(E) + \cdots = 2\pi\hbar\left(n + \frac{1}{2}\right),$$

where $S_0(E) = S(E)$ is the classical action as above, and the higher-order terms are ***quantum corrections*** to the classical action; the corrections are given by more complicated integrals, involving higher powers of the momentum, and derivatives of the potential.

Unfortunately this series is basically always divergent, and even the method of Borel summation cannot be directly applied. When we try to apply our trick

$$\sum_{k=0}^{\infty} S_k(E)\hbar^k = S_0(E) + \int_0^\infty \sum_{k=0}^{\infty}\left(\frac{S_{k+1}(E)t^k}{k!}\right)e^{-t/\hbar}dt,$$

we find that the integrand has infinitely many poles along the positive real axis, so the series is not Borel summable for positive values of E, which are precisely the ones that we care about.

In 1969, Bender and Wu published a beautiful paper [3], dealing with the case in which

$$V = \tfrac{1}{2}x^2 + \lambda x^4$$

is the potential of a ***quartic anharmonic oscillator***. They analyzed the energy levels as functions of λ (rather than $\hbar$), and made several remarkable discoveries that explain why the relevant series expansions must diverge. Using a mixture of approximate and exact methods, they found strong evidence that the energy levels, when viewed as functions of the parameter λ, have infinitely many branch points in a neighbourhood of $\lambda = 0$. Moreover, different energy levels can be connected to one another by a process of analytic continuation, in which λ is allowed to take on complex values and travel along a loop encircling the branch points. Further contributions were made by Balian–Bloch, Écalle and Voros [13], and eventually these ideas were treated rigorously by Aoki–Kawai–Takei [1, 2] and Delabaere–Dillinger–Pham [5], using the theory of resummation/resurgence.

5 Quantum field theory

Predictions for quantities in high-energy physics are typically given by an integral of the form

$$\int_M f(\phi)e^{iS(\phi)/\hbar}d\phi \tag{6}$$

where M is a space of "fields". This space is usually an infinite-dimensional manifold, although in gauge theories this space also has singularities. The laws of physics are encoded in the structure of the ***action functional***

$$S : M \to \mathbb{R},$$

that appears in the exponential, while the factor f in the integrand is chosen according to which observable quantity we are trying to compute, e.g. energy, location of a particle, etc.

This integral is not mathematically well-defined, due to the lack of appropriate measures on infinite-dimensional manifolds. But a technique known as perturbative expansion (or Feynman expansion), gives us a series

$$\int_M f(\phi) e^{iS(\phi)/\hbar} d\phi = \sum_{k=0}^{\infty} a_k g^k$$

for this ill-defined quantity in terms of a small parameter g, called the coupling constant. This constant indicates the strength of the forces through which particles interact.

Making sense of the coefficients in the perturbative series is already difficult; they at first appear to be infinite, but then these infinities are removed by the infamous processes of "renormalization" and "regularization". The first few terms can then be computed, and excellent agreement with experiment is found.

But even once these techniques have been applied to define the coefficients in the series, a major problem remains: the series is basically always divergent, and it is often not even Borel summable. So there remains no rigorous definition of the quantity we are actually trying to compute. This divergence can be seen directly in many examples, but there is also an elegant physical argument for why this ought to be the case, put forward by Dyson in a classic two-page paper [8].

One might conclude, as Dyson does, that this means that perturbation theory is incomplete, and a proper definition of the integral (6) must be given by some other mathematical means, or even a new theory of physics developed. But it is also conceivable that despite the divergence, the coefficients in the perturbative expansion actually contain all the information that is necessary for the complete reconstruction of the value of (6). There is evidence that, at least in many toy models, this second point of view has some merit. For example, it has been used for rigorous constructions of some field theories such as low-dimensional ϕ^4 (e.g. [9, 10]). Many recent physics papers have suggested that resurgence provides a way to uncover nonperturbative effects from the perturbative expansion; see, for example, the recent surveys [6, 7, 11].

6 Plan for the course

The lectures will be based mostly on the books of Candelpergher–Nosmas–Pham [4] and Shatalov–Sternin [12]. We aim to cover the following topics:

- **Basic asymptotics:** Asymptotic series, examples
- **Integral transforms:** Borel and Laplace transforms, convolution integrals
- **Analytic continuation:** Endless continuability, infinite-sheeted Riemann surfaces
- **Singularities of functions:** Encoding singularities of functions in sheaves on the circle, interaction with Laplace transform and convolution, variation operators
- **Resurgence:** Resurgent functions and resurgent symbols, summability, Stokes operators and alien derivatives
- **Applications:** To be determined. Possibilities include
  - Écalle's work on normal forms for dynamical systems
  - The "exact" WKB method and the quantum quartic oscillator
  - Recent studies in nonperturbative quantum field theory

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